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Dynamic Modeling of Rotary Double Inverted Pendulum Using Classical Mechanics

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Abstract: *Modeling of nonlinear systems always play a vital role in the design of advanced control system techniques. An accurate model is the success of any type of control system design. Pendulum systems are well known and good bench mark demonstrations of automatic control techniques. They are nonlinear, unstable, non-minimum phase systems which showcase modern control methods. In this paper the full dynamics of the Parallel Rotary Double Inverted Pendulum (RDIP) are derived using classical mechanics and Lagrangian formulation.*

Keywords: *Rotary Double Inverted Pendulum, Modeling, Classical Mechanics.*

1. Introduction

A variety of inverted pendulum designs are available in the literature. The system configuration depends on two factors, the method of actuation and the number of degrees of freedom. There were two types of actuation reviewed, linear and rotary. The simplest controllable inverted pendulum would consist of a pendulum link directly coupled to a motor shaft must have at least two degrees of freedom, one for the position of the base of the pendulum and the other for the pendulum angle [1]. For higher degrees of freedom, either more single degree of freedom links are added, or the existing links are allowed to move in multiple dimensions. Rotary Double Inverted Pendulum (RDIP) as shown in Figure.1, an extended version of Rotary Inverted Pendulum is considered for the further discussions. Considering unequal masses and stabilizing the system with respect to the center of gravity which is an actual need in aircraft, ship and automobile systems, this can easily be simulated using RDIP. The control objective is to balance the pendulums in the upright position.

Many engineering applications need a compact and accurate description of the dynamic behavior of the system under consideration. This is especially true of automatic control applications. Dynamic models describing the system of interest can be constructed using the first principles of physics, chemistry, biology, and so forth [2].

In this paper the dynamics of the RDIP are derived using the rotational geometry [3] of the system which is explained in section II. Many papers have only considered the rotational inertia of the pendulum for a single principal axis or neglected it altogether [4]. The system dynamics for Rotary Double Inverted Pendulum with a full inertia tensor using a Lagrangian formulation are presented in this paper. Also a linearized model is obtained by neglecting the disturbance torques. Simulation results show the open loop system characteristics.

2. Mathematical Modeling of Rotary Inverted Pendulum

2.1 Fundamentals

The physical structure of the RDIP is as shown in Figure 1 with two pendulums of different lengths which accounts of different masses on both sides, where the related physical parameters of the system are listed in the

Table 1. The DC motor is used to apply a torque to the rotating arm and the link between rotating arm and Pendulum arm 1 & 2 is not actuated but free to rotate. The horizontal arm output is angular displacement α . Initially two pendulums will be in the pendant position. ie q1=q2=0^{o}. As the shaft rotates, the rotating beam also gets rotated. So a centrifugal force will be developed at both ends. This force is converted to the force which rotates the pendulums to the inverted position.

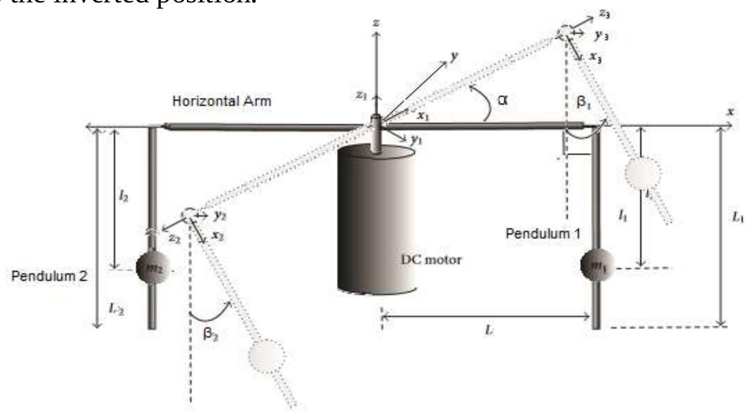

Fig. 1: Schematic of Rotary double inverted pendulum system..

TABLE I: Definition of parameters related to rotary double inverted pendulum

Parameter	Definition	Value
M_1	Centre of mass of pendulum 1	0.25 Kg
M_2	Centre of mass of pendulum 2	0.13 Kg
L_1	Distance from joint to Centre of mass of pendulum1	0.24 m
L_2	Distance from joint to Centre of mass of pendulum2	0.13 m
M	Mass of the rotating arm	0.52 Kg
L	Length of the rotating arm	0.215 m
β_1	Pendulum1 angle with respect to inertial axis.	rad
β_2	Pendulum2 angle with respect to inertial axis.	rad
α	Angular displacement of rotating arm	rad
b_0, b_1, b_2	Viscous coefficients of rotating arm, pendulum1and pendulum2 respectively	0.004, 0.0031, 0.000088 M-ms
τ_1	Torque experienced by rotating arm	N-m
τ_2 , τ_3	Disturbance torque experienced by pendulum1 and pendulum2	N-m
K_m	Torque constant	0.005 N.m/A
K_b	Back emf constant	0.001 Volt/rad
R	Resistance of motor circuit	2 Ω

A right hand coordinate system has been used to define the rotation of horizontal arm and the two pendulum arms. The coordinate axes of the rotating arm and two pendulum arms are the principal axes such that the inertia tensor are diagonal in the form as in (1).

$$J_1 = \begin{pmatrix} J_{x1x} & 0 & 0 \\ 0 & J_{y1y} & 0 \\ 0 & 0 & J_{z1z} \end{pmatrix} \quad J_2 = \begin{pmatrix} J_{x2x} & 0 & 0 \\ 0 & J_{y2y} & 0 \\ 0 & 0 & J_{z2z} \end{pmatrix} \quad J3 = \begin{pmatrix} J_{x3x} & 0 & 0 \\ 0 & J_{y3y} & 0 \\ 0 & 0 & J_{z3z} \end{pmatrix} \quad (1)$$

The angular rotation of rotating arm α, is measured in the horizontal plane where a counter clockwise direction (when viewed from above) is positive [4]. The angular rotation of pendulum 1, β_1, is measured in the vertical plane where a counterclockwise direction (when viewed from the front) is positive, same as in pendulum 2 is β_2, when pendulums are hanging down in the stable equilibrium position $\beta_1 = \beta_2 = 0$.

The dynamic model of the RDIP is derived based on the following assumptions,

- The motor shaft and rotating arm are assumed to be rigidly coupled and infinitely stiff;
- Pendulums are assumed to be infinitely stiff;
- The coordinate axes of rotating arm and pendulums are the principal axes such that the inertia tensors are diagonal
- The motor rotor inertia is assumed to be negligible. However, this term may be easily added to the moment of inertia of rotating arm
- Only viscous damping is considered. All other forms of damping (such as Coulomb) have been neglected

2.2 Lagrangian Formulation Using Classical Mechanics

The Lagrangian L, of a dynamical system is a mathematical function that summarizes the dynamics of the system. For a simple mechanical system, it is the value given by the kinetic energy of the particle minus the potential energy of the particle but it may be generalized to more complex systems. It is used primarily as a key component in the Euler Lagrange equation. In classical mechanics, the natural form of the Lagrangian is defined as the Kinetic Energy, *T*, of the system minus its Potential Energy, *V*,

$$L=T-V$$

Rotation matrix for rotating arm and pendulums are defined as follows. Rotation matrix from the base to the rotating arm is given by,

$$R_1 = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

The rotation matrix from rotating arm to the pendulum1 is given by (multiplying a diagonal matrix that maps the frame 1 to frame 2 with the rotation matrix of β_1)

$$R_2 = \begin{pmatrix} \cos(\beta_1) & \sin(\beta_1) & 0 \\ -\sin(\beta_1) & \cos(\beta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sin(\beta_1) & -\cos(\beta_1) \\ 0 & \cos(\beta_1) & \sin(\beta_1) \\ 1 & 0 & 0 \end{pmatrix} \tag{3}$$

Similarly rotation matrix from rotating arm to the pendulum2 is given by

$$R_3 = \begin{pmatrix} \cos(\beta_2) & \sin(\beta_2) & 0 \\ -\sin(\beta_2) & \cos(\beta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sin(\beta_2) & -\cos(\beta_2) \\ 0 & \cos(\beta_2) & \sin(\beta_2) \\ -1 & 0 & 0 \end{pmatrix} \tag{4}$$

Velocities of the arms are defined as:

The angular velocity of horizontal arm is given by $\omega_1 = [0 \quad 0 \quad \alpha^{\bullet}]^T$ (5)

Initially the base frame is considered to be rest so that the joint between base and rotating arm is also at rest

$$V_0 = [0 \quad 0 \quad 0]^T$$

Velocity of frame of reference is given is by

$$\omega_1 \times [L \quad 0 \quad 0]^T$$

Therefore the total linear velocity of the horizontal arm is given by

$$V_1 = V_0 + \omega_1 \times [L \quad 0 \quad 0]^T \quad = [0 \quad L\dot{\alpha} \quad 0]^T \tag{6}$$

Angular velocity of pendulum 1 is given by

$$\omega_2 = [0 \quad 0 \quad \beta^{\bullet}{}_1]^T + R_2\omega_1 = [-\cos(\beta_1)\dot{\alpha} \quad \sin(\beta_1)\dot{\alpha} \quad \dot{\beta}_1]^T \tag{7}$$

Total linear velocity of the centre of mass of pendulum1 is given by

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$$V_2 = R_2(\omega_1 \times [L \quad 0 \quad 0]^T) + \omega_2 \times [l_1 \quad 0 \quad 0]^T = \begin{pmatrix} \dot{\alpha} L \sin(\beta_1) \\ \dot{\alpha} L \cos(\beta_1) + l_1 \dot{\beta}_1 \\ -\dot{\alpha} l_1 \sin(\beta_1) \end{pmatrix} \tag{8}$$

Where velocity of joint between rotating arm and pendulum1 in reference frame is given by

$$\omega_1 \times [L \quad 0 \quad 0]^T$$

Which in reference frame 2 gives

$$R_2(\omega_1 \times [L \quad 0 \quad 0]^T)$$

Angular velocity of pendulum 2 is given by

$$\omega_3 = [0 \quad 0 \quad \dot{\beta}_2]^T + R_3\omega_1 = [-\dot{\alpha}\cos(\beta_2) \quad \dot{\alpha}\sin(\beta_2) \quad \dot{\beta}_2]^T \tag{9}$$

Total Linear velocity of pendulum 2(as in the case of pendulum1) is given by

$$V_3 = R_3(\omega_1 \times [L \quad 0 \quad 0]^T) + \omega_3 \times [l_2 \quad 0 \quad 0]^T = \begin{pmatrix} \dot{\alpha} L \sin(\beta_2) \\ \dot{\alpha} L \cos(\beta_2) + l_2 \dot{\beta}_2 \\ -\dot{\alpha} l_2 \sin(\beta_2) \end{pmatrix} \tag{10}$$

Energies of the arms are defined as:
For rotating arm,
Potential energy is given by $P_0 = 0$
Kinetic energy is,

$$k_0 = \frac{1}{2}(V_1^T m V_1 + \omega_1^T J_1 \omega_1) = \frac{1}{2}\dot{\alpha}^2 (mL^2 + Jz_1z) \tag{11}$$

Potential energy of pendulum1 is given by

$$P_1 = m_1 g l_1 \cos(\beta_1) \tag{12}$$

And Kinetic energy is,

$$k_1 = \frac{1}{2}(V_2^T m_1 V_2 + \omega_2^T J_2 \omega_2) \quad = \frac{1}{2} m_1(\dot{\alpha}^2 L^2 \sin^2(\beta_1) + (\dot{\alpha} L\cos(\beta_1) + l_1\dot{\beta}_1)^2 \quad + \dot{\alpha}^2 l_1^2 \sin^2(\beta_1)) + \frac{1}{2}(\dot{\alpha}^2 Jx_2x\cos^2(\beta_1) \tag{13}$$

$$+ \dot{\alpha}^2 \sin^2(\beta_1) Jy_2y + \dot{\beta}_1^2 Jz_2z)$$

Potential energy of pendulum 2 is given by

$$P_2 = m_2 g l_2 \cos(\beta_2) \tag{14}$$

Kinetic energy is given by

$$K_2 = \frac{1}{2}(V_3^T m_2 V_3 + \omega_3^T J_3 \omega_3) = \frac{1}{2} m_2(\dot{\alpha}^2 L^2 \sin^2(\beta_2) + (\dot{\alpha} L\cos(\beta_2) + l_2\dot{\beta}_2)^2 + \dot{\alpha}^2 l_2^2 \sin^2(\beta_2))$$

$$+ \frac{1}{2}(\dot{\alpha}^2 Jx_3x\cos^2(\beta_2) + \dot{\alpha}^2 \sin^2(\beta_2) Jy_3y + \dot{\beta}_2^2 Jz_3z) \tag{15}$$

Total potential energy is $P_{total} = P_0 + P_1 + P_2$ and total kinetic energy is $K_{total} = K_0 + K_1 + K_2$
The mathematical equation can be derived using Lagrangian, L=KE-PE, where KE is the kinetic energy and PE is the potential energy. The Lagrangian equation can be written as,

$$\frac{d}{dt}(\partial L / \partial \dot{q}_i) + b_i \dot{q}_i - \frac{\partial L}{\partial q_i} = Q_i \tag{16}$$

Where,
$q_i = [\alpha \quad \beta_1 \quad \beta_2]^T$ is the generalised coordinate, $b_i = [b_0 \quad b_1 \quad b_2]^T$ is generalised viscous damping coefficient. $Q_i = [\tau_1 \quad \tau_2 \quad \tau_3]^T$ Be the generalised force (torque including disturbance torque).

We can write the Lagrange function as L=KE-PE as follows,

$$L=\frac{1}{2}\dot{\alpha}^2(mL^2+Jz_1z)+\frac{1}{2}m_1(\dot{\alpha}^2L^2\sin^2(\beta_1)+(\dot{\alpha}L\cos(\beta_1)+l_1\dot{\beta}_1)^2+\dot{\alpha}^2l_1^2\sin^2(\beta_1))+\frac{1}{2}(\dot{\alpha}^2Jx_2x\cos^2(\beta_1)+\dot{\alpha}^2\sin^2(\beta_1)Jy_2y+\dot{\beta}_1^2Jz_2z)$$
$$+\frac{1}{2}m_2(\dot{\alpha}^2L^2\sin^2(\beta_2)+(\dot{\alpha}L\cos(\beta_2)+l_2\dot{\beta}_2)^2+\dot{\alpha}^2l_2^2\sin^2(\beta_2))+\frac{1}{2}(\dot{\alpha}^2Jx_3x\cos^2(\beta_2)+\dot{\alpha}^2\sin^2(\beta_2)Jy_3y+\dot{\beta}_2^2Jz_3z)-m_2gl_2\cos(\beta_2)$$
$$-m_2gl_2\cos(\beta_2) \quad (17)$$

By using (16) and (17) dynamic equation can be written as,

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}\begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta}_1 \\ \ddot{\beta}_2 \end{pmatrix}+\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}=\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \quad (18)$$

Where,

$$\begin{aligned} A_{11} &= mL^2+J_1+m_1L^2+m_1l_1^2\sin^2(\beta_1)+m_2L^2 \\ &+m_2l_2^2\sin^2(\beta_2)+J_2\sin^2(\beta_1)+J_3\sin^2(\beta_2) \quad A_{12}=m_1l_1L\cos(\beta_1) \quad A_{13}=m_2l_2L\cos(\beta_2) \\ A_{21} &= m_1l_1L\cos(\beta_1) \quad A_{22}=(m_1l_1^2+J_2) \quad A_{23}=0 \quad A_{31}=m_2l_2L\cos(\beta_2) \\ A_{32} &= 0 \quad A_{33}=(m_2l_2^2+J_3) \end{aligned} \quad (19)$$

$$\begin{aligned} A_1 &= -m_1l_1L\sin(\beta_1)\dot{\beta}_1^2+m_1l_1^2\dot{\alpha}\dot{\beta}_1\sin(2\beta_1)-m_2l_2L\sin(\beta_2)\dot{\beta}_2^2+m_2l_2^2\dot{\alpha}\dot{\beta}_2\sin(2\beta_2) \\ &+J_2\dot{\alpha}\dot{\beta}_1\sin(2\beta_1)+J_3\dot{\alpha}\dot{\beta}_2\sin(2\beta_2)+b_0\dot{\alpha} \\ A_2 &= -\frac{1}{2}m_1l_1^2\dot{\alpha}^2\sin(2\beta_1)-m_1l_1g\sin(\beta_1)+b_1\dot{\beta}_1-\frac{1}{2}J_2\dot{\alpha}^2\sin(2\beta_1) \\ A_3 &= -\frac{1}{2}m_2l_2^2\dot{\alpha}^2\sin(2\beta_2)-m_2l_2g\sin(\beta_2)+b_2\dot{\beta}_2-\frac{1}{2}J_3\dot{\alpha}^2\sin(2\beta_2) \end{aligned}$$

Since the arms are long and slender moment of inertia is considered to be negligible, more over the arms have rotational symmetry such that the moment of inertia in the principal axes are equal thus inertia tensor can be approximated as follows ,

$$J_1=\begin{pmatrix} J_{x1x} & 0 & 0 \\ 0 & J_{y1y} & 0 \\ 0 & 0 & J_{z1z} \end{pmatrix}\approx\begin{pmatrix} 0 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_1 \end{pmatrix} \quad J_2=\begin{pmatrix} J_{x2x} & 0 & 0 \\ 0 & J_{y2y} & 0 \\ 0 & 0 & J_{z2z} \end{pmatrix}\approx\begin{pmatrix} 0 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_2 \end{pmatrix} J_3=\begin{pmatrix} J_{x3x} & 0 & 0 \\ 0 & J_{y3y} & 0 \\ 0 & 0 & J_{z3z} \end{pmatrix}\approx\begin{pmatrix} 0 & 0 & 0 \\ 0 & J_3 & 0 \\ 0 & 0 & J_3 \end{pmatrix} \quad (20)$$

Above dynamic equation can be written in a little easier way by making the following substitution:

Total moment of inertia of rotating arm about the pivot point,

$$\hat{J}_1=J_1+mL^2 \quad (21)$$

Moment of inertia of pendulum1 about its pivot point,

$$\hat{J}_2=J_2+m_1l_1^2 \quad (22)$$

Moment of inertia of pendulum2, $\hat{J}_3=J_3+m_2l_2^2$ (23)

Total moment of inertia experience by the motor when pendulum 1 and 2 are in hanging position,

$$\hat{J}_0=\hat{J}_1+m_1L^2+m_2L^2=J_1+mL^2+m_1L^2+m_2L^2 \quad (24)$$

From (18) the control input is the torque τ_1 applied to the pivot of the rotating arm. DC motor is used to drive the rotating arm, so the voltage is taken as the control input. As far as we are neglecting the effect of inductor, the torque and the voltage can be related by the equation,

$$\tau=\frac{K_mV}{R}-\frac{K_mK_b\dot{\alpha}}{R} \quad (25)$$

Where K_m, K_b, R are the motor parameters mentioned in the Table 1. A compact form can be obtained by substituting (21)-(25) into (18), thus obtaining a coupled electro mechanical equation as follows,

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta}_1 \\ \ddot{\beta}_2 \end{pmatrix} + \begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = \begin{pmatrix} \frac{K_m}{R} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} V \\ \tau_2 \\ \tau_3 \end{bmatrix} \tag{26}$$

Where,

$$\begin{aligned} A_{11} &= \hat{J}_0 + m_1 l_1^2 \sin^2(\beta_1) + m_2 l_2^2 \sin^2(\beta_2) \\ &\quad + J_2 \sin^2(\beta_1) + J_3 \sin^2(\beta_2) \qquad A_{12} = m_1 l_1 L \cos(\beta_1) \qquad A_{13} = m_2 l_2 L \cos(\beta_2) \\ A_{21} &= m_1 l_1 L \cos(\beta_1) \qquad A_{22} = \hat{J}_2 \qquad A_{23} = 0 \qquad A_{31} = m_2 l_2 L \cos(\beta_2) \qquad A_{32} = 0 \qquad A_{33} = \hat{J}_3 \\ A_1' &= -m_1 l_1 L \sin(\beta_1) \dot{\beta}_1^2 + m_1 l_1^2 \dot{\alpha} \dot{\beta}_1 \sin(2\beta_1) - m_2 l_2 L \sin(\beta_2) \dot{\beta}_2^2 + m_2 l_2^2 \dot{\alpha} \dot{\beta}_2 \sin(2\beta_2) \\ &\quad + J_2 \dot{\alpha} \dot{\beta}_1 \sin(2\beta_1) + J_3 \dot{\alpha} \dot{\beta}_2 \sin(2\beta_2) + (b_0 + \frac{K_m K_b}{R}) \dot{\alpha} \\ A_2' &= -\frac{1}{2} \dot{\alpha}^2 \sin(2\beta_1) \hat{J}_2 - m_1 l_1 g \sin(\beta_1) + b_1 \dot{\beta}_1 \qquad A_3' = -\frac{1}{2} \dot{\alpha}^2 \sin(2\beta_2) \hat{J}_3 - m_2 l_2 g \sin(\beta_2) + b_2 \dot{\beta}_2 \end{aligned} \tag{27}$$

2.3 Linearized state equation

Linearized equation of the dynamic system for the two equilibrium position: upright and downward are derived below.

2.3.1 Upright position

Finding the linearized model using the equilibrium point where $\beta_1 = \beta_2 = 180°$. After expanding the Taylor series at x=π

$$\text{Sin}(x) \approx \pi - x, \ \sin^2(x) \approx 0, \ \cos(x) \approx -1$$

We can make approximation of the nonlinear equation in (26) around the equilibrium point $\beta_1 = \beta_2 = 180°$ thus equation became,

$$\begin{aligned} &\hat{J}_0(\ddot{\alpha}) - m_1 l_1 L \ddot{\beta}_1 - m_2 l_2 L \ddot{\beta}_2 + (b_0 + \frac{K_m K_b}{R}) \dot{\alpha} = \frac{K_m V}{R} - m_1 l_1 L(\ddot{\alpha}) + \hat{J}_2 \ddot{\beta}_1 + m_1 l_1 g(\beta_1 - \pi) + b_1 \dot{\beta}_1 = \tau_2 \\ &-m_2 l_2 L(\ddot{\alpha}) + \hat{J}_3 \ddot{\beta}_2 + m_2 l_2 g(\beta_2 - \pi) + b_2 \dot{\beta}_2 = \tau_3 \end{aligned} \tag{28}$$

Here x can be taken as x=$[\alpha \quad \beta_1 - \pi \quad \beta_2 - \pi \quad \dot{\alpha} \quad \dot{\beta}_1 \quad \dot{\beta}_2]^T$ be the state variable and control input be the voltage, v. Then (28) can arrange in the form of

$$\begin{aligned} P\dot{x} &= Qx + Ru \\ \dot{x} &= P^{-1}Qx + P^{-1}Ru =: Ax + Bu \end{aligned} \tag{29}$$

Where,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{J}_0 & -m_1 l_1 L & -m_2 l_2 L \\ 0 & 0 & 0 & -m_1 l_1 L & \hat{J}_2 & 0 \\ 0 & 0 & 0 & -m_2 l_2 L & 0 & \hat{J}_3 \end{pmatrix} Q = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -(b_0 + \frac{K_m K_b}{R}) & 0 & 0 \\ 0 & -m_1 g l_1 & 0 & 0 & -b_1 & 0 \\ 0 & 0 & -m_2 g l_2 & 0 & 0 & -b_2 \end{pmatrix} \text{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{K_m}{R} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Disturbance torque may be neglected in the analysis, so the R matrix became

$$R = [0 \quad 0 \quad 0 \quad \frac{K_m}{R} \quad 0 \quad 0]^T$$

2.3.2 Downward position

In downward position the operating points are, $\beta_1 = \beta_2 = 0°$ From the Taylors series, at the small values of x,

$$x^2 \approx 0, \quad \sin(x) \approx x, \quad \sin^2(x) \approx 0$$

The nonlinear equation in (26) may be approximated around the equilibrium point $\beta_1 = \beta_2 = 0°$. Thus equation becomes,

$$\begin{aligned} &\hat{J}_0(\ddot{\alpha}) + m_1 l_1 L \ddot{\beta}_1 + m_2 l_2 L \ddot{\beta}_2 + \left(b_0 + \frac{K_m K_b}{R}\right)\dot{\alpha} = \frac{K_m V}{R} \; m_1 l_1 L(\ddot{\alpha}) + \hat{J}_2 \ddot{\beta}_1 - m_1 l_1 g(\beta_1) + b_1 \dot{\beta}_1 = \tau_2 \\ &m_2 l_2 L(\ddot{\alpha}) + \hat{J}_3 \ddot{\beta}_2 - m_2 l_2 g(\beta_2) + b_2 \dot{\beta}_2 = \tau_3 \end{aligned} \tag{30}$$

Here x can be taken as x= $[\alpha \quad \beta_1 \quad \beta_2 \quad \dot{\alpha} \quad \dot{\beta}_1 \quad \dot{\beta}_2]^T$ be the state variable and. Equation (30) can rearrange in the form of

$$\begin{aligned} &P_1 \dot{x} = Q_1 x + R_1 u \\ &\dot{x} = P_1^{-1} Q_1 x + P_1^{-1} R_1 u =: A_1 x + B_1 u \end{aligned} \tag{31}$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{J}_0 & m_1 l_1 L & m_2 l_2 L \\ 0 & 0 & 0 & m_1 l_1 L & \hat{J}_2 & 0 \\ 0 & 0 & 0 & m_2 l_2 L & 0 & \hat{J}_3 \end{pmatrix} Q_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\left(b_0 + \frac{K_m K_b}{R}\right) & 0 & 0 \\ 0 & m_1 g l_1 & 0 & 0 & -b_1 & 0 \\ 0 & 0 & m_2 g l_2 & 0 & 0 & -b_2 \end{pmatrix} R_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{K_m}{R} & 0 & 0 \end{bmatrix}^T$$

3. Simulation Results and Discussions

Consider the parameters of the RDIP system as shown in Table 1. Simulation of the derived model is carried with these parameters. The zero input response (ZIR) of the system near the equilibrium point 180^0 is shown in Figure 2. Since the open loop response of the system is shown in Figure 2 the pendulums will settle at the stable equilibrium point (0, 0) which indicates that the system is tracing the dynamics as per the modelling. The response of the rotating arm for the same is shown in Figure 3.

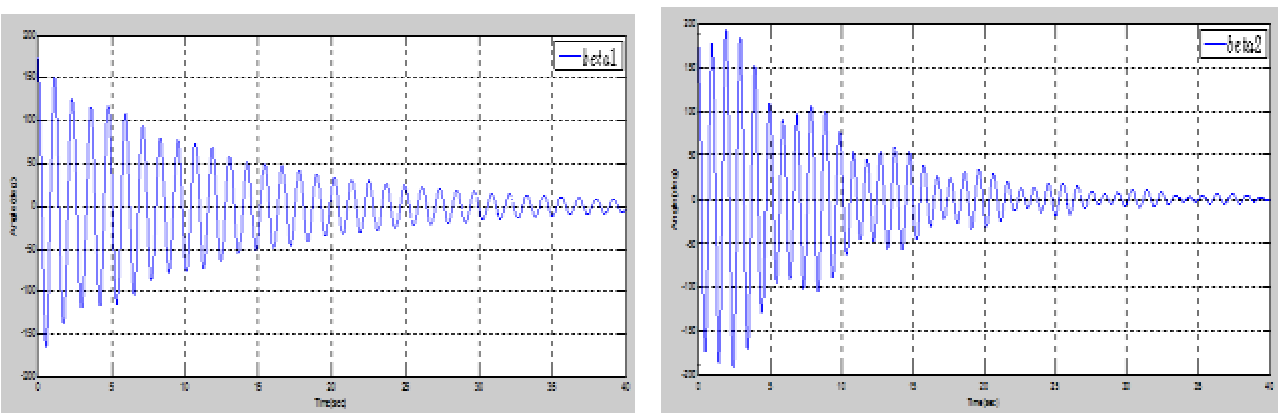

Fig. 2: Zero input response of β_1 and β_2 for the initial condition β_1 (0)= 175^0 and β_2 (0) = 178^0 .

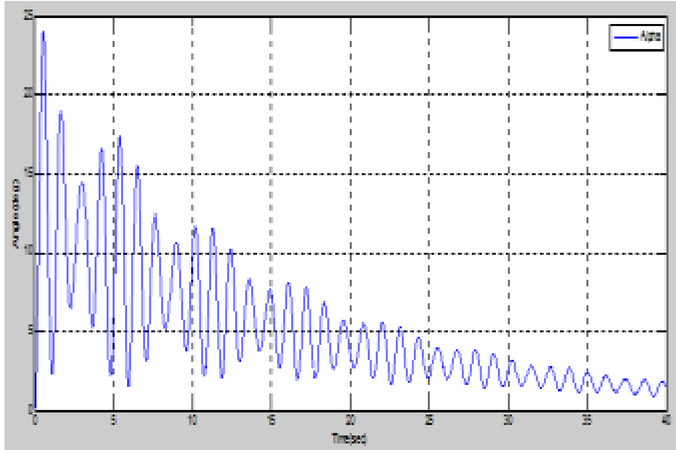

Fig. 3: Zero input response of rotating arm

4. Conclusion

The full dynamics of the rotary double inverted pendulum is obtained using classical mechanics and Lagrangian formulation. Linearized equations for both the upright and downward positions of both pendulums have been derived, as well as the coupled motor pendulum equations. It also observes that the derived model has the same general form of the pendulums. The results may be compared with the experimental data and suitable control law may be applied to stabilize the system as the future scope.

5. References

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