ISBN 978-93-84422-56-1
Proceedings of International Conference on Architecture, Structure and Civil Engineering (ICASCE'15)
Antalya (Turkey) Sept. 7-8, 2015 pp. 1-7

Analysis of Axially Loaded Non-prismatic Beams with General End Restraints Using Differential Quadrature Method

Mahmoud E. Said[1]
[1]Faculty of Engineering, Cairo University, Giza, Egypt

Abstract: *Differential quadrature method (DQM) is used to study the stability behavior and free vibration of axially loaded tapered beams with elastic end restraints. The governing differential equation is discretized over the considered domain at N finite sampling points yielding a system of N algebraic equations in N unknowns. Then the boundary conditions are discretized and merged into governing differential equation which transforming the problem to an eigenvalue one in two parameters (axial load and natural frequency). However, assigning a value for one of the two parameters, the other can be obtained. The obtained solutions were verified against the FEM solution and found in close agreement. The effects of different parameters related to the studied model on the load and frequency parameters will be investigated.*

Keywords: *Tapered beams, end restraints, differential quadrature, axial load and natural frequencies.*

1. Introduction

Beams are characterized by their profile (the shape of their cross-section), their length, and their material. The weight of the structural elements have a great influence on justify the functional requirements of the structure, therefore optimization of weight is needed for architectural or economical requirement. To optimize the weight of structural elements non-prismatic configurations are commonly used such as tapered beams. Beams are also described by how they are supported. Supports restrict lateral and/or rotational movements so as to satisfy stability conditions as well as to limit the deformations to a certain allowance.

It is very hard problem to derive a closed form solutions to represent the structural behavior of non-uniform elements under static and dynamic loads. In whatever way, deriving a closed form solution needs an idealization process to simplify the mathematical treatments which misrepresent the physical problems. The idealization of the normal end conditions (hinged – fixed – free) where differ from reality support actions, therefore translational and/or rotational end restraints are used to consider the actual situation of the support movements in the analysis. Analytical solutions for simple cases of prismatic and non-prismatic elements with elastic end restraints to obtain both stability and vibration behavior of structural elements are found in literature [1-3].

Taha M.H. and Abohadima, S. [4] studied the free vibration of non-uniform shear beam resting on elastic foundation using Bessel functions. Ruta [5] used the Chebychev series to study the vibration of non-prismatic beam. Sato [6] studied the effect of end restraints and axial force on the vibration frequencies for tapered beams using Ritz method. Another numerical methods such as finite element method [7-8] and differential quadrature method (DQM) [9-12] are used to study certain configurations of such models.

In the present paper, the stability and vibration behavior of axially loaded tapered beams with lateral translation and/or rotational end restraints are investigated using the DQM. Beam depth is assumed to increase linearly from one end till the midpoint of the beam and then to decrease linearly to the other end, whereas the width of the beam is considered constant. This problem is solved before using FEM [4], therefore the obtained solutions were verified against the FEM solution and found in close agreement. The main differences between the present work and the previous one are the method of solution, the implementation of lateral translation end restraints and the employee of axial load. The effects of different parameters related to the studied model on the load and frequency parameters are investigated.

2. Problem Formulation

2.1. Vibration Equation

Consider a linearly tapered beam with variable depth d(x), width (b) and length (L) is subjected to an axial force (Po) with elastic end rotational and lateral translation restraints as shown in Fig.(1).

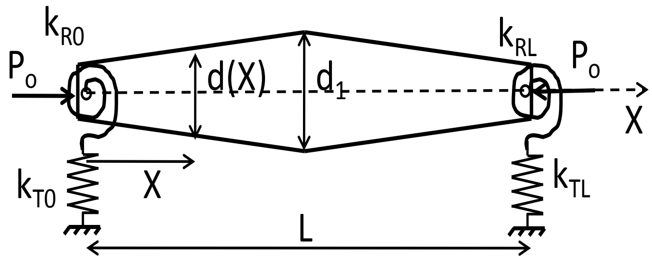

Fig. 1: Axially loaded tapered column with elastic end restraints

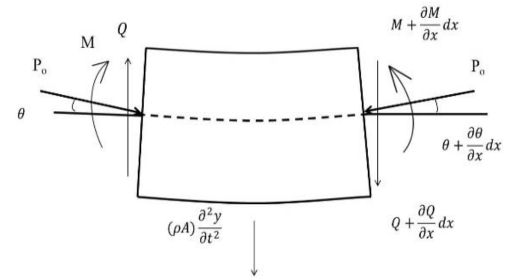

Fig. 2: Forces acting on beam differential element

Where k_{Ro} is the stiffness of rotational restraint at x=0; k_{RL} is the stiffness of rotational restraint at x=L; $k_{To}$is the stiffness of lateral translation restraint at x=0; $k_{TL}$is the stiffness of lateral translation restraint at x=L; Q (x) is the internal shear force; M (x) is internal moment; ρ is density of the beam per unit volume; A(x) is area of the beam cross section at distance x; I(x) is moment of inertia of the beam cross section at distance x; E is the young's modulus of the beam material.

The free vibration equation for the shown non-prismatic beam is given as:

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right)+P_o\frac{\partial^2 y}{\partial x^2}+\rho A(x)\frac{\partial^2 y}{\partial t^2}=0 \tag{1}$$

Using dimensionless parameters X=*x*/L & Y=*y*/L, the partial differential equation (PDE) becomes:

$$\frac{\partial^2}{\partial X^2}\left(\frac{\mathrm{EI(X)}}{\mathrm{L}^3}\frac{\partial^2 \mathrm{W}}{\partial \mathrm{X}^2}\right)+\left(\frac{\mathrm{P_o}}{\mathrm{L}}\right)\frac{\partial^2 \mathrm{W}}{\partial \mathrm{X}^2}+\rho\mathrm{A(x)L}\frac{\partial^2 \mathrm{W}}{\partial t^2}=0 \tag{2}$$

For solving this linear PDE eqn. (2), first separation of variables was applied where the lateral displacement is distributed by two independent functions, one for spatial variation (mode shape function) and the other for time variation. Then applied DQM to transform the governing PDE into a homogeneous system of N algebraic equations solved numerically. However, the solution of the linear version of eqn. (2) depends on the boundary conditions at the beam ends.

Using separation of variable method, the solution of the PDE (eqn. 2) may be assumed as:

$$\mathrm{W(X,t)}=\Phi(\mathrm{X})\Psi(\mathrm{t}) \tag{3}$$

Where $\Phi(\mathrm{X})$is the linear mode function, $\Psi(\mathrm{t})$is a function representing the variation with time. Substituting eqn. (3) into eqn. (2), eqn. (2) transform to eqn. (4), where ω = separation constant.

$$\frac{d^2}{dX^2}\left(EI(X)\frac{d^2\Phi}{dX^2}\right)+L^2P_o\frac{d^2\Phi}{dX^2}-\rho A(X)L^4\Phi(X)\omega^2=0 \tag{4}$$

$$\ddot{\Psi}+\omega^2\Psi(t)=0 \tag{5}$$

For initial conditions $\Psi(0)=1$ and $\dot{\Psi}(0)=0$, The solution of eqn.(5) equals $\Psi(t)=cos\omega t$.

For the tapered beam shown in Fig.(1) where the depth of the beam increases linearly from d_o at X=0 to d_1 at X=0.5, then decreases lineally to d_o at X=1, where the tapering ratio α is defined as $\alpha=d_I/d_o$,then the area and the moment of inertia at distance x are given as:

$$A(X)=\begin{cases}A_o[1+(\alpha-1)*2X]\,; & 0\le X\le {}^1\!/_2\\ A_o[1+(\alpha-1)*2*(1-X)]; & {}^1\!/_2\le X\le 1\end{cases} \tag{6}$$

$$I(X)=\begin{cases}I_o[1+(\alpha-1)*2X]^3; & 0\le X\le {}^1\!/_2\\ I_o[1+(\alpha-1)*2*(1-X)]^3; & {}^1\!/_2\le X\le 1\end{cases} \tag{7}$$

Substitution eqn. (6) and eqn. (7) into eqn. (4) gets the governing equation:

$$\frac{d^4\Phi}{dX^4}+(\beta)\frac{d^3\Phi}{dX^3}+(\eta_1+\eta_2P_o)\frac{d^2\Phi}{dX^2}+(-\xi_2\omega^2)\Phi(x)=0 \tag{8}$$

Where $\beta = \frac{2}{I(X)}\frac{dI(X)}{dX}$; $\eta_1 = \frac{1}{I(X)}\frac{d^2I(X)}{dX^2}$; $\eta_2 = \frac{L^2}{EI(X)}$; and $\xi_2 = \frac{\rho AL^4}{EI(X)}$

2.2. Boundary Conditions

The dimensionless lateral translation and rotational elastic end restraints at $X = 0$ are related to the derivatives of lateral displacement at the beam ends as: (since $\Phi = \Phi(\mathrm{X})$.)

$$k_{To}\Phi = -\frac{d}{dX}\left(\frac{EI(X)}{L^3}\frac{d^2\Phi}{dX^2}\right) \tag{9}$$

$$k_{Ro}\frac{d\Phi}{dX} = \frac{EI_o}{L}\frac{d^2\Phi}{dX^2} \tag{10}$$

Similarly the dimensionless elastic end restraints at $X = 1$ are expressed as:

$$k_{TL}\Phi = \frac{d}{dX}\left(\frac{EI(X)}{L^3}\frac{d^2\Phi}{dX^2}\right) \tag{11}$$

$$k_{RL}\frac{d\Phi}{dX} = -\frac{EI_o}{L}\frac{d^2\Phi}{dX^2} \tag{12}$$

For the shown tapered beam, using eqn. (6) and eqn. (7); one can get the boundary condition as:

$$k_{To}\Phi = -\frac{EI_o}{L^3}[\Phi''' - 6(1-\alpha)\Phi''] \tag{13}$$

$$k_{Ro}\Phi' = \frac{EI_o}{L}\Phi'' \tag{14}$$

$$k_{TL}\Phi = \frac{EI_o}{L^3}[\Phi''' + 6(1-\alpha)\Phi''] \tag{15}$$

$$k_{RL}\Phi' = -\frac{EI_o}{L}\Phi'' \tag{16}$$

3. Problem Solution Using DQM

3.1. Differential Quadrature Method (DQM)

The solution of eqn. (8) is obtained using the DQM, where the solution domain is discretized into N sampling points and the derivatives at any point are approximated by a weighted linear summation of all the functional values at the other points [12].

$$\left.\frac{d^m f(x)}{d^m x}\right|_{x_i} \approx \sum_{j=1}^{N} C_{i,j}^{m} \,.\, f(x_j), \quad for\ i = 1,2,\dots,N\ and\ m = 1,2,\dots,M \tag{17}$$

Where M is the order of the highest derivative in the governing equation, $f(x_j)$ is the functional value at point of $x=x_j$ and $C_{i,j}^m$ are the weighting coefficients relating the derivative m at $x=x_i$ to the functional value at $x=x_j$. To get the weighting coefficients, many polynomials with different base functions can be used. Lagrange interpolation formula is used, where the functional value at a point x is approximated by all the functional values $f(x_k)$, (k=1, N) as:

$$f(x) = \sum_{k=1}^{N} \frac{\prod_{j=1}^{N}(x-x_j)}{(x-x_k)\,.\,\prod_{k=1}^{N}(x_i-x_k)} f(x_k) \qquad i = 1,2,\dots,N\ \&\ k = 1,2,\dots,N \tag{18}$$

Substitution of eqn. (18) into eqn.(17) gets the weighting coefficients of the first derivative as [12]:

$$C_{i,j}^{(1)} = \frac{\prod_{k=1}^{N}(x_i-x_k)}{(x_i-x_j)\prod_{k=1}^{N}(x_j-x_k)} \quad for\ (i \neq j)\ and\ (i,j = 1,N) \tag{19}$$

$$C_{i,j}^{(1)} = -\sum_{j=1, j\neq i}^{N} C_{i,j}^{1} \quad for\ (i = j)\ and\ (i,j = 1,N) \tag{20}$$

Applying the chain rule on eqn.(17), the weighting coefficients of the (m) order expresses as:

$$C_{i,k}^{(m)} = -\sum_{j=1, j\neq i}^{N} C_{i,k}^{(1)} C_{i,k}^{(m-1)} \quad for\ (i,j = 1,N)\ and\ (m = 1,M) \tag{21}$$

As the DQM is a numerical method, its accuracy is affected by both the number and the distribution of discretization points. In boundary value problems, it is found that the irregular distribution of the discretization points with smaller mesh spaces near the boundary to cope the steep variation near the boundaries is more adaptable. One of the frequently used distributions for mesh points is the normalized Gauss-Chebychev – Lobatto distribution given as:

$$x_i = \frac{1}{2}\left[1 - cos\left(\frac{(i-1)\pi}{N-1}\right)\right]\ ,\ i = 1,2,\dots,N \tag{22}$$

3.2.Discretization of Boundary Condition

The boundary condition at beam ends may be written in the DQM discretized form as:

$$k_{To}\Phi_1 = -\frac{EI_o}{L^3}\left[\sum_{k=1}^{N} C_{1,k}^{(3)}\Phi_k - 6(1-\alpha)\sum_{k=1}^{N} C_{1,k}^{(2)}\Phi_k\right] \quad (23)$$

$$k_{Ro}\sum_{k=1}^{N} C_{1,k}^{(1)}\Phi_k = \frac{EI_o}{L}\sum_{k=1}^{N} C_{1,k}^{(2)}\Phi_k \quad (24)$$

$$k_{TL}\Phi_N = \frac{EI_o}{L^3}\left[\sum_{k=1}^{N} C_{N,k}^{(3)}\Phi_k - 6(1-\alpha)\sum_{k=1}^{N} C_{N,k}^{(2)}\Phi_k\right] \quad (25)$$

$$k_{RL}\sum_{k=1}^{N} C_{N,k}^{(1)}\Phi_k = -\frac{EI_o}{L}\sum_{k=1}^{N} C_{N,k}^{(2)}\Phi_k \quad (26)$$

These equations cannot be easily substituted directly into the governing equation, so one can couple these four equations together to give four expressions for the unknown functional values Φ_1; Φ_2; Φ_{N-1} and Φ_N in terms of the other functional values ϕ_i, (i=3, N-2).

$$\Phi_1 = \sum_{i=3}^{N-2}\mu_1\Phi_i\ ,\quad \Phi_2 = \sum_{i=3}^{N-2}\mu_2\Phi_i\ ,\quad \Phi_3 = \sum_{i=3}^{N-2}\mu_3\Phi_i\ ,\quad \Phi_4 = \sum_{i=3}^{N-2}\mu_1\Phi_4 \quad (27)$$

Where μ's are numerical coefficients and Φ's are required unknowns.

3.3.Discretization of Governing Equation

Using the DQM, the governing equation of can be discretized at N-4 sampling points as:

$$\sum_{k=1}^{N} C_{i,k}^{(4)}\Phi_k + \beta(X_i)\sum_{k=1}^{N} C_{i,k}^{(3)}\Phi_k + [\eta_1(X_i)+\eta_2(X_i)P_o]\sum_{k=1}^{N} C_{i,k}^{(2)}\Phi_k - \xi_2(X_i)\omega^2\Phi_i = 0 \quad (28)$$

Then, using Lagrange interpolation polynomial and then obtains:

$$\sum_{k=3}^{N-2}\left(C_{1_k} + \beta_k C_{2_k} + \eta_{1_k} + \eta_{2_k}P_o - \left(\xi_{2_k}\omega^2\right)\delta_{ik}\right)C_{3_k}\Phi_k = 0 \quad (29)$$

Where C_1, C_2, C_3 are parameters introduced to simplify the obtained equation, and

$$\delta_{ik} = kronecr\ delta\ \begin{cases}1 & i=k\\ o & i\neq k\end{cases} \quad (30)$$

Eqn. (28) represents a homogeneous system of N-4 equations with two parameters (P_o and ω). Assigning a value for one of the two parameters leads to Eigenvalue problem, which can be solved to obtain the value of the other parameter. However, to calculate the critical axial load P_{cr}, the ω is assumed zero to eliminate the inertia term in governing equation. For ω calculations, appropriate value for axial load ($P_o<P_{cr}$) is assumed. A MATLAB program has been designed to solve the non-dimensional system eqn. (28) and calculating critical loads, vibration natural frequencies and the functional values of dimensionless lateral displacement at different locations along the beam.

3.4.Verification Of Present Solution

Introducing dimensionless parameters for both of restraint stiffness at beam ends:

$$\overline{K}_{To} = \frac{k_{To}L^3}{EI_o},\ \ \overline{K}_{Ro} = \frac{k_{Ro}L}{EI_o},\ \ \overline{K}_{TL} = \frac{k_{TL}L^3}{EI_o},\ \ \overline{K}_{RL} = \frac{k_{RL}L}{EI_o} \quad (31)$$

It is found that the value $\overline{K}_R$=10^3 is sufficient to simulate the clamping condition and $\overline{K}_R$=0 represent the pinned condition. On the other hand, free lateral translation can represent when $\overline{K}_T$ =10^{-1}, while to simulate constraining lateral translation, large value for lateral restraint stiffness are assumed ($\overline{K}_T$=10^5).

3.4.1. Comparison With Analytical Results For Prismatic Beam

To verify the present solution, values of the frequency parameter $\lambda = \sqrt[4]{\frac{\rho A_o\omega^2L^4}{EI_o}}$ for different end restraints of prismatic beam are calculated and compared with exact solutions [4] in Table (1).

TABLE I: values of frequency parameter for prismatic beam

λ	Pinned-Pinned	Pinned-Clamped	Clamped-Clamped
Exact Solution	3.141	3.926	4.730
DQM	3.141	3.922	4.717

To obtain the critical load P_{cr}, the eigenvalue problem is solved assuming zero natural frequency (ω). The values of stability parameter $\lambda_b = \sqrt{\frac{P_{cr}L^2}{EI_o}}$ for the case of prismatic beam with conventional end restraints calculated from the DQM is close to the exact values as shown in Table (2).

TABLE II: values of Stability parameter for prismatic beam

stability parameter (λ_b)	Pinned-Pinned	Pinned-Clamped	Clamped-Clamped
Exact Solution	$\pi = 3.1415$	$1.431\pi = 4.4956$	$2\pi = 6.283$
DQM	3.1413	4.4898	6.2643

3.4.2. Comparison with FEM Results for Tapered Beam

To verify the present solution in the static case, values of the frequency parameter λ for different configurations of beam are calculated and compared with those obtained from the finite element analysis FEM [4] in Table (3). The values shown in the table indicate close agreement between present analysis and those obtained using the FEM for low tapering ratio α ($1 < \alpha < 1.2$). There are some deviations in results of large values of tapering ratio α ($1.3 < \alpha < 1.5$).

TABLE III: values of frequency parameter for tapered beam

End Restrains Stiffness				Tapering Ratio : $\alpha = d_1/d_o$						Analysis
$\bar{K}_{To}$	$\bar{K}_{TL}$	$\bar{K}_{Ro}$	$\bar{K}_{RL}$	1.0	1.1	1.2	1.3	1.4	1.5	
10^5	10^5	0	0	3.141	3.248	3.349	3.449	3.534	3.620	FEM
				3.141	3.284	3.393	3.478	3.542	3.591	DQM
10^5	10^5	0.1	0.1	3.173	3.276	3.373	3.466	3.554	3.638	FEM
				3.172	3.315	3.426	3.511	3.578	3.629	DQM
10^5	10^5	1	1	3.399	3.479	3.557	3.634	3.708	3.780	FEM
				3.398	3.544	3.662	3.758	3.838	3.905	DQM
10^5	10^5	10	10	4.156	4.201	4.247	4.294	4.341	4.388	FEM
				4.155	4.321	4.468	4.600	4.721	4.833	DQM

4. Numerical Results

To obtain the critical load P_{cr} (buckling load), the eigenvalue problem is solved assuming the natural frequency (ω) equal to zero. The loading ratio is defined as $\bar{P}_o = \frac{P_o}{\bar{P}_{cr}} = \frac{P_o L^2}{\pi^2 E I_o}$, where Euler critical load for simply supported prismatic beam is $\bar{P}_{cr} = \frac{\pi^2 E I_o}{L^2} =$

Values for λ_b are calculated for different values of tapering ratio $\alpha = d_1/d_o$ and different values of end restraints stiffness $(\bar{K}_T, \bar{K}_R)$ are presented in Fig. (3) to Fig. (6). It is observed that the stability parameter increases as the overall stiffness of the beam increases. However, the overall stiffness of the beam consists of the beam flexural stiffness and the stiffness of elastic end restraints. In light of the overall stiffness, the variation in stability parameter value shown in the following figures can be qualitatively interpreted.

In Fig. (3) variation of λ_b with α and $\bar{K}_{RL}$ ($\bar{K}_{Ro}$=0) observed that for $\bar{K}_{RL} = 0$ the gain in stability parameter is about 44.2% and for $\bar{K}_{RL} = 10^3$ the gain in stability parameter is about 85.1% due to increasing of α value from 1.0 to 1.5. Fig (4) presents the variation of λ_b with α and $\bar{K}_{RL}$ ($\bar{K}_{Ro}$=10^3). It is observed that for $\bar{K}_{RL} = 0$ the gain in stability parameter is about 85.1% and for $\bar{K}_{RL} = 10^3$ the gain in stability parameter is about 100% due to increasing of α value from 1.0 to 1.5. Fig. (5) displays the variation of λ_b with α and $\bar{K}_{TL}$ ($\bar{K}_{To}$=10^5, $\bar{K}_{Ro} = \bar{K}_{RL} = 0$). It is observed that for $\bar{K}_{TL} = 0$ the gain in stability parameter is about 7% and for $\bar{K}_{TL} = 10^5$ the gain in stability parameter is about 44.2% due to increasing of α value from 1.0 to 1.5. Fig. (6) shows the variation of λ_b with α and $\bar{K}_{TL}$ ($\bar{K}_{To}$=10^5, $\bar{K}_{Ro} = \bar{K}_{RL} = 10^3$). It is observed that for $\bar{K}_{TL} = 0$ the gain in stability parameter is about 100% and for $\bar{K}_{TL} = 10^5$ the gain in stability parameter is about 100% due to increasing of α value from 1.0 to 1.5

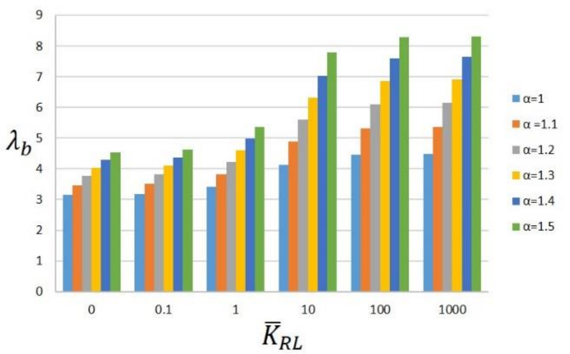

Fig. 3: Variation of λ_b with α and $\overline{K}_{RL}$ ($\overline{K}_{Ro} = 0$, $\overline{K}_{To} = \overline{K}_{TL} = 10^5$)

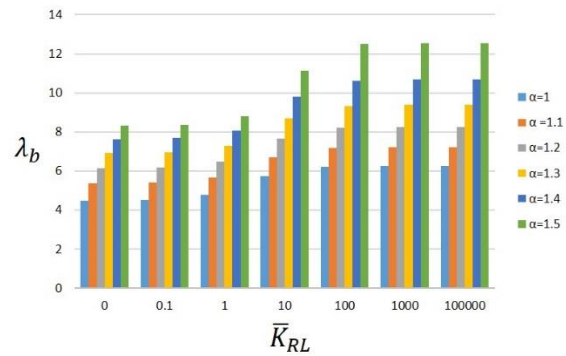

Fig. 4: Variation of λ_b with α and $\overline{K}_{RL}$ ($\overline{K}_{Ro}=10^3$, $\overline{K}_{To} = \overline{K}_{TL} = 10^5$)

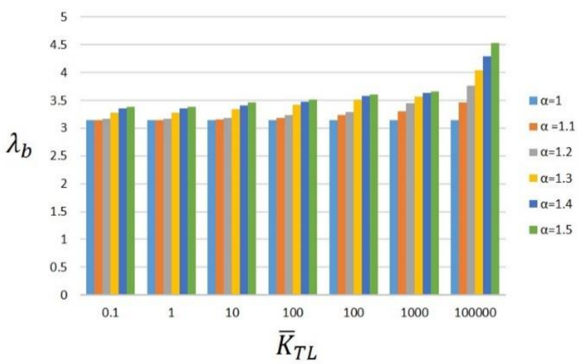

Fig. 5: Variation of λ_b with α and $\overline{K}_{TL}$ ($\overline{K}_{To} = 10^5$, $\overline{K}_{Ro} = \overline{K}_{RL} = 0$)

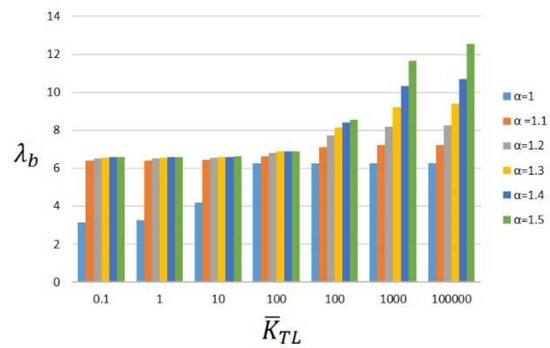

Fig. 6: Variation of λ_b with α and $\overline{K}_{TL}$ ($\overline{K}_{To}=10^5$, $\overline{K}_{Ro} = \overline{K}_{RL} = 10^5$)

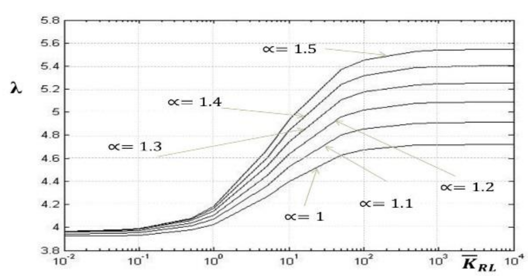

Fig. 7: Influence of $\overline{K}_{RL}$ on λ for different ∝ $\overline{K}_{To} = \overline{K}_{TL} = 10^5$; $\overline{K}_{Ro} = 10^3$; $\overline{P}_o = 0$

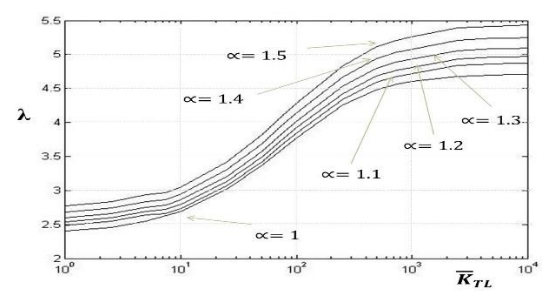

Fig. 8: Influence of $\overline{K}_{TL}$ on λ for different ∝**,** $\overline{K}_{To} = 10^5$; $\overline{K}_{Ro} = \overline{K}_{RL} = 10^3$; $\overline{P}_o = 0$

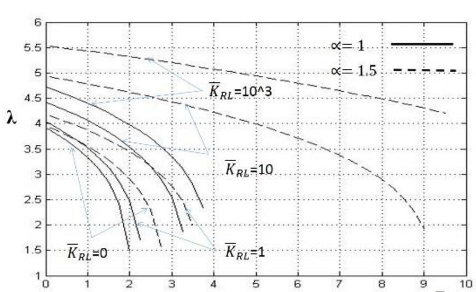

Fig. 9: Influence of $\overline{K}_{RL}$ on λ for different $\overline{P}_o$ $\overline{K}_{To} = \overline{K}_{TL} = 10^5$; $\overline{K}_{Ro} = 10^3$

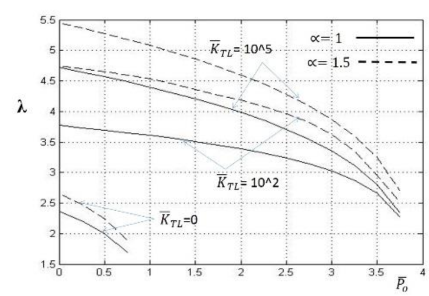

Fig. 10: *nfluence of the Influence of* $\overline{K}_{TL}$ *on* λ *for different* $\overline{P}_o$, $\overline{K}_{To} = 10^5$; $\overline{K}_{Ro} = \overline{K}_{RL} = 10^3$

The influences of different paramters on the frequency parameter are shown in Fig. (7) to Fig. (10). The effects of restraints stiffness parameter at one support ($\overline{K}_{RL}$)and tapering ratio (α) on the frequency parameter

(λ) are shown in Figure (7). It is observed that as the restraint stiffness parameter increases, the frequency parameter increases. This is due to the increase in the total beam stiffness. The increase in frequency parameter is more pronounced for large values of $\overline{K}_{RL}$. In Fig. (8), the lateral translational stiffness parameter($\overline{K}_{TL}$) at one end is investigated. It is clear that, the frequency parameter increases as the translational stiffness increase as the system stiffness increases. The effect of axial loading ratio $\overline{P}_o$ on the frequency parameter is depicted in Fig. (9) and Fig. (10). Fig. (9) shows the effect of rotational stiffness at one end and Fig. (10) shows the influence of translational stiffness at one end. It obvious that as the loading ratio increases, the frequency parameter decreases.

5. Conclusions

Based on the numerical analysis presented in the previous study, it can be demonstrated that the differential quadrature method is an efficient method in solving the free vibration and stability of tapered beams with translational and rotational end restraints subjected to initial compression axial load with a high accuracy using a small number of sampling points. It is concluded that both the stability parameter and the frequency parameter of tapered beam increases as the overall stiffness of the system increases. The overall stiffness of the system increases as the tapering ratio increases, and as the stiffness of the elastic end restraints increase. The influence of tapering ratio is more significant for small values of the tapering ratio and for flexible system. Asymptotic values of end restraints parameters equivalent to conventional supporting conditions are ($\overline{K}_T$=1000 and $\overline{K}_R$ =0) for simply support condition and ($\overline{K}_T$=1000 and $\overline{K}_R$=1000) for clamped condition.

6. References

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