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Dynamics of Exponential Functionally Graded Timoshenko Beams with Single Delamination

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***Abstract*:** *This paper presents a theoretical investigation for the dynamics of a delaminated functionally graded beam (FGB) with single arbitrary delamination. It is assumed that material properties vary along the beam thickness only according to exponential distributions. The Timoshenko beam theory and the 'free mode' and 'constrained mode' assumption in delamination vibration are adopted. A rather new analytical solution is developed for exponential FGM beams with clamped-clamped, clamped-free, clamped-hinged and hinged-hinged boundary conditions. The effects of various parameters, such as delamination size and location, material properties on the vibration of the beam are studied in detail. These results provide useful information in the study of the free vibration of delaminated FGM.*

Keywords: *Delaminated beam; Free Vibration; Analytical Solution; Functionally Graded Material.*

1. Introduction

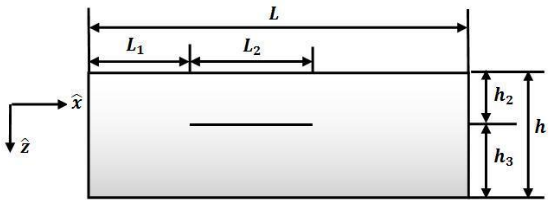
Advances in material synthesis technologies have spurred the development of a new class of materials, called functionally graded materials (FGMs), with various applications in aerospace, transportation, energy, electronics and biomedical engineering [1]. Further, the increasing use of beams as structural components in various fields such as civil, aerospace and mechanical engineering has necessitated the study of their dynamic characteristics.

Many studies have been conducted on the dynamic behavior of intact FGM beams [2-6], but dynamic analysis of the delaminated FGM beam has received limited attention. Recently, Liu and Shu [7] and Liu et al. [8] presented analytical solutions for the free vibration analysis of the FGM beams with single delamination based on the Bernoulli-Euler beam theory. Literature review shows that although there are quite a few papers presenting dynamic analyses of delaminated FGM beams based on the Bernoulli-Euler theory, no work investigating the vibration behavior of delaminated FGM Timoshenko beam has been reported. It is worth to mention that the classical Euler-Bernoulli beam theory is only able to predict the frequencies of flexural vibration of the lower modes of thin beams with adequate precision.

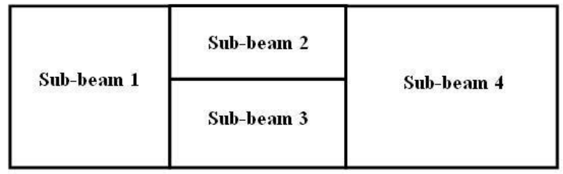
In the present study, rather a new analytical solution is developed to study the free vibration of exponential functionally graded beams with a single delamination. Timoshenko beam theory, the 'free mode' and 'constrained mode' assumption in delamination vibration are adopted. This is the first study on the effects of delamination geometrical parameters on the natural frequency of exponentially functionally graded beams.

2. Problem Definition

Consider a FGM beam of length L and constant thickness h, with single through the width delamination (See Fig. (1a)). As shown in Fig. (2a), after delamination, a representative beam can be viewed as a combination of four beams connected at the delamination boundaries $x = L_1$ and $x = L_1 + L_2$.



(a) delaminated FGM



(b) sub-beams 1-4

Fig. 1: Geometry of a delaminated beam: (a) delaminated beam and (b) four interconnected sub-beams.

In this way, we will have four sub-beams of 1 to 4 with lengths and thicknesses of $L_i \times h_i$ $(i = 1-4)$, where $L_2 = L_3$, $L_4 = L-(L_1+L_2)$, $h_1 = h_4 = h$, and h_2 and h_3 are the thicknesses of sub-beams 2 and 3, respectively. It should be said that because of the existence of the delamination, the sub-beams, especially in a delaminated segment, may no longer remain thin in the beam configuration even though the original perfect beam may be thin. To simulate the "open" and "closed" behaviors between the delaminated surfaces, we use the piecewise-linear spring model as proposed by Luo and Hanagud [9]. The spring stiffness is then set to be equal to zero (0) for the free mode and infinity (∞) for the constrained mode.

It is assumed that the Young's modulus, the Poisson's ratio and mass density of the beam vary continuously only in the thickness direction ($\hat{z}$ -axis) with exponential function as follows:

$$E(\hat{z}) = E_0 e^{\beta_1 \hat{z}}, \;\; \rho(\hat{z}) = \rho_0 e^{\beta_2 \hat{z}}, \;\; \nu(\hat{z}) = \nu_0 e^{\beta_3 \hat{z}} \tag{1}$$

where E_0, ρ_0 and ν_0 are the values of the Young's modulus, mass density and Poisson's ratio at the midplane ($\hat{z} = 0$) of the beam. Also, β_i $(i = 1-3)$ is a constant defining the material property variation along the thickness direction, and $\beta_i = 0$ corresponds to an isotropic homogeneous beam.

3. Problem Formulations

Based on the Timoshenko beam theory, the displacement fields of an arbitrary point in the i^{th} beam, denoted by $u_i\left(\hat{x}, \hat{z}, t\right)$ and $w_i\left(\hat{x}_i, \hat{z}_i, t\right)$, respectively, take the form of:

$$\begin{aligned} u_i(\hat{x}_i, \hat{z}_i, t) &= u_{0i} + \hat{z}_i \psi_i(\hat{x}_i, t) \\ w_i(\hat{x}_i, \hat{z}_i, t) &= w_{0i}(\hat{x}_i, t) \qquad (\mathrm{i} = 1-4) \end{aligned} \tag{2}$$

where $u_{0i}\left(\hat{x}_i, t\right)$, $w_{0i}\left(\hat{x}_i, t\right)$ and $\psi_i(\hat{x}_i, t)$ are the axial displacement, flexural displacement and bending rotation of the i^{th} beam and $\hat{x}_i$ is the axial coordinate of the sub-beam whose origin is located at the left boundary of each sub-beam. By definition, we can write:

$$\varepsilon_i = \varepsilon_{0i} + \hat{z}_i \kappa_i = \frac{\partial u_{0i}}{\partial \hat{x}_i} + \hat{z}_i \frac{\partial \psi_i}{\partial \hat{x}_i}, \qquad \gamma_i = \frac{\partial w_{0i}}{\partial \hat{x}_i} \tag{3}$$

in which ε_{0i}, κ_i and γ_i are the normal strain, flexural curvature and shear strain of the i^{th} sub-beam. The kinetic energy T and the strain energy U of the vibrating delaminated FGM beam can be written as:

$$\begin{aligned} T &= \sum_{i=1}^{4} \frac{1}{2} \int_0^{L_i} \left\{ I_{1i} w_{0i,t}{}^2 + I_{3i} \psi_{i,t}{}^2 \right\} d\hat{x}_i \\ U &= \sum_{i=1}^{4} \frac{1}{2} \int_0^{L_i} \left\{ N_{\hat{x}i} \varepsilon_{0i} + M_{\hat{x}i} \kappa_i + Q_i \gamma_i \right\} b d\hat{x}_i + \frac{1}{2} \int_0^{L_2} k \left(w_2 - w_3 \right)^2 d\hat{x}_2 \end{aligned} \tag{4}$$

in which [7]:

$$\begin{aligned} N_{\hat{x}i} &= A_{11i} \varepsilon_{0i} + B_{11i} \kappa_i \\ M_{\hat{x}i} &= B_{11i} \varepsilon_{0i} + D_{11i} \kappa_i \\ Q_i &= A_{55i} \gamma_i \end{aligned} \tag{5}$$

where M_i, Q_i and N_i are the bending moment, shear and axial forces, respectively, and:

$$\left(A_{11i}, B_{11i}, D_{11i}\right)=\int_{-h_i/2}^{h_i/2} E(\hat{z})\ (1, \hat{z}_i, \hat{z}_i^2) d\hat{z}_i, \qquad A_{55i}=\int_{-h_i/2}^{h_i/2} k_s G(\hat{z})\ d\hat{z}_i$$
$$\left(I_{1i}, I_{3i}\right)=\int_{-h_i/2}^{h_i/2} \rho(\hat{z})\ (1, \hat{z}_i^2) d\hat{z}_i \tag{6}$$

where k_s is the beam cross-sectional shape factor, and k is the spring stiffness. It should be mentioned that in all above relations, the symbol "," used as a subscript stands for the differentiation with respect to any variable followed after it. The above relations for the kinetic and potential energies will be used in subsequent section to form the functional. Note that the axial inertia term is neglected in the kinetic energy.

4. Analytical Solution

In the present work, Ritz method combined with the Lagrange multipliers are used to study the free vibration characteristics of the delaminated FGM beam. The main advantage of the Lagrange multiplier technique is that the choice of the assumed displacement functions is easy because they do not have to satisfy the boundary conditions of the problem. The simple Legendre polynomials are chosen as displacement functions, and this simplifies the problem further in that orthogonality properties of these polynomials lead to simple energy expression.

Harmonic solutions for the variables $w_{0i}(\hat{x},t)$, $\psi_i(\hat{x},t)$ and $u_i(\hat{x},t)$ are assumed as:

$$w_{0i}(\hat{x},t)=W_i(\hat{x})e^{i\omega t}, \quad \psi_i(\hat{x},t)=\Psi_i(\hat{x})e^{i\omega t}, \ u_i(\hat{x},t)=U_i(\hat{x})e^{i\omega t} \qquad (i=1,2,3,4) \tag{7}$$

in which $W_i(\hat{x}), \Psi_i(\hat{x})$ and $U_i(\hat{x})$ are the displacement functions of the i^{th} sub-beam and ω is the circular frequency. As mentioned above, the displacement functions can be expressed in terms of the simple Legendre polynomials and are given by:

$$W_i(x)=\sum_{n=0}^{N} W_{in}P_n(x), \qquad \Psi_i(x)=\sum_{n=0}^{N} \Psi_{in}P_n(x), \qquad U_i(x)=\sum_{n=0}^{N} U_{in}P_n(x) \tag{8}$$

Here, P_n is the simple Legendre polynomial of degree n and the axial coordinate $\hat{x}$ is transformed to the interval $-1 \le x \le 1$ by letting $x=\dfrac{\hat{x}-L_i/2}{L_i/2}$.

We have six boundary conditions (B.C.s) for sub-beams 1 and 4, i.e. three boundary conditions for each sub-beam. These six boundary conditions which are not satisfied by the assumed series are imposed as constraints. For common boundary conditions, these constraints can be written as follows:

Clamped-Clamped Beam (CC):

$$\begin{aligned} &W_1(-1)=0,\ \Psi_1(-1)=0, U_1(-1)=0 \\ &W_4(1)=0,\ \Psi_4(1)=0, U_4(1)=0 \end{aligned} \tag{9-a}$$

Clamped-Hinged Beam (CH):

$$\begin{aligned} &W_1(-1)=0,\ \Psi_1(-1)=0, U_1(-1)=0 \\ &W_4(1)=0,\ M_4(1)=0, U_4(1)=0 \end{aligned} \tag{9-b}$$

Hinged-Hinged Beam (HH):

$$\begin{aligned} &W_1(-1)=0,\ M_1(-1)=0, U_1(-1)=0 \\ &W_4(1)=0,\ M_4(1)=0, U_4(1)=0 \end{aligned} \tag{9-c}$$

Clamped- Free Beam (CF):

$$\begin{aligned} &W_1(-1)=0,\ \Psi_1(-1)=0, U_1(-1)=0 \\ &Q_4(1)=0,\ M_4(1)=0,\ \ N_4(1)=0 \end{aligned} \tag{9-d}$$

in which prime denotes differentiation with respect to *x*. Furthermore, we have 18 continuity and compatibility conditions (C.C.s) at delamination boundaries as follows:

at the left end of sub-beams 2 and 3:

$$
\begin{aligned}
&W_1(1) = W_2(-1), \quad W_1(1) = W_3(-1), \quad \Psi_1(1) = \Psi_2(-1), \quad \Psi_1(1) = \Psi_3(-1) \\
&U_1(1) - e_2\Psi_1(1) = U_2(-1), \qquad U_1(1) + e_3\Psi_1(1) = U_3(-1) \\
&Q_1(1) = Q_2(-1) + Q_3(-1), \qquad N_1(1) = N_2(-1) + N_3(-1) \\
&M_1(1) - M_2(-1) - M_3(-1) + e_2 N_2(-1) - e_3 N_3(-1) = 0
\end{aligned}
\tag{10}
$$

at the right end of sub-beams 2 and 3:

$$
\begin{aligned}
&W_2(1) = W_4(-1), \quad W_3(1) = W_4(-1), \quad \Psi_2(1) = \Psi_4(-1), \quad \Psi_3(1) = \Psi_4(-1) \\
&U_4(-1) - e_2\Psi_1(-1) = U_2(1), \qquad U_4(-1) + e_3\Psi_1(-1) = U_3(1) \\
&Q_4(-1) = Q_2(1) + Q_3(1), \qquad N_4(-1) = N_2(1) + N_3(1) \\
&M_4(1) - M_2(1) - M_3(1) + e_2 N_2(1) - e_3 N_3(1) = 0
\end{aligned}
\tag{11}
$$

Boundary, continuity and compatibility conditions in terms of assumed series are presented in Appendix A. A variational principle is formulated based on the kinetic and strain energies by a procedure similar to one followed by Washizu [10]. This variational principle along with the constraint conditions is used to solve the vibration problem. The functional to be extremized is given by the expression:

$$
F = T + U - \sum_{i=1}^{6} \alpha_i \,(\text{B.C.s}) - \sum_{i=1}^{9} \beta_i \,(\text{C.C.s at x=L}_1) - \sum_{i=1}^{9} \gamma_i \,(\text{C.C.s at x=L}_1\text{+L}_2)
\tag{12}
$$

where $\alpha_i \ (i = 1,...,6), \beta_i$ and $\gamma_i \ (i = 1,2,...,9)$ are the Lagrange multipliers. Substituting the assumed series for $W_i(x), \Psi_i(x)$ and $U_i(x) \left(i = 1,2,3,4\right)$ in equation (12) and simplifying yields:

$$
\begin{aligned}
F = \sum_{i=1}^{4} \Bigg\{ & -\frac{\omega^2 b L_i}{4} \sum_{n=0}^{N} \frac{2}{2n+1}\left(I_{1i} W_{in}^{\ 2} + I_{3i}\Psi_{in}^{\ 2}\right) + \frac{A_{11i} b}{L_i} \int_{-1}^{1} \sum_{n=1}^{N} U_{in} \sum_{k_1=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (2n-4k_1-1) P_{n-2k_1-1}(x) \sum_{m=1}^{N} U_{im} \sum_{k_2=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (2m-4k_2-1) P_{m-2k_2-1}(x)\, dx + \\
& \frac{2B_{11i} b}{L_i} \int_{-1}^{1} \sum_{n=1}^{N} U_{in} \sum_{k_1=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (2n-4k_1-1) P_{n-2k_1-1}(x) \sum_{m=1}^{N} \Psi_{im} \sum_{k_2=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (2m-4k_2-1) P_{m-2k_2-1}(x)\, dx + \frac{D_{11i} b}{L_i} \int_{-1}^{1} \sum_{n=1}^{N} \Psi_{in} \sum_{k_1=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (2n-4k_1-1) P_{n-2k_1-1}(x) \sum_{m=1}^{N} \Psi_{im} \sum_{k_2=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (2m-4k_2-1) P_{m-2k_2-1}(x)\, dx + \\
& \frac{A_{55i} b}{L_i} \int_{-1}^{1} \sum_{n=1}^{N} W_{in} \sum_{k_1=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (2n-4k_1-1) P_{n-2k_1-1}(x) \sum_{m=1}^{N} W_{im} \sum_{k_2=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (2m-4k_2-1) P_{m-2k_2-1}(x)\, dx + A_{55i} b \int_{-1}^{1} \sum_{n=0}^{N} \Psi_{in} P_n(x) \sum_{m=1}^{N} W_{im} \sum_{k_2=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (2m-4k_2-1) P_{m-2k_2-1}(x)\, dx + \frac{A_{55i} b L_i}{4} \sum_{n=0}^{N} \frac{2}{2n+1} \Psi_{in}^{\ 2} \Bigg\} \\
& + \frac{k L_2}{4} \sum_{n=0}^{N} \frac{2}{2n+1}\left(W_{2n}^{\ 2} + W_{3n}^{\ 2} - 2W_{2n} W_{3n}\right) - \sum_{i=1}^{6} \alpha_i \,(\text{B.C.s}) - \sum_{i=1}^{9} \beta_i \,(\text{C.E.s at x=L}_1) - \sum_{i=1}^{9} \gamma_i \,(\text{C.E.s at x=L}_1 + L_2)
\end{aligned}
\tag{13}
$$

In calculating the functional F, we have used the following properties of Legendre polynomial [11]:

$$
P_n'(x) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(2n - 4k - 1\right) P_{n-2k-1}(x) \qquad (n \geq 1)
$$

The necessary extremizing conditions are given by:

$$
\frac{\partial F}{\partial W_{in}} = \frac{\partial F}{\partial \Psi_{in}} = \frac{\partial F}{\partial U_{in}} = 0, \quad \left(\text{n=0,1,2,...}\right) \text{ and } \left(\text{i=1,2,3,4}\right)
\tag{14}
$$

Using equation (14) in conjunction with equation (13) results in a system of linear algebraic equations which, in matrix form, can be written as:

$$
[A]\left\{q_1, q_2, q_3, q_4\right\}^T = [B]
\tag{15}
$$

in which the right hand side of equation (15) consists of Lagrange multipliers and:

$$
q_i = \left\{W_{i0}, W_{i1}, ..., W_{in}, \Psi_{i0}, \Psi_{i1}, ..., \Psi_{in}, U_{i0}, U_{i1}, ..., U_{in}\right\}
$$

Solving equation (15) for W_{in}, Ψ_{in} and $U_{in} \quad (i = 1,2,3,4)$ and substituting into the equations (9-11) results in a system of homogenous linear algebraic equations with the Lagrange multipliers as unknowns. The system of equations is given by:

$$
[C]\left\{\alpha_1, \alpha_2, ..., \alpha_6, \beta_1, \beta_2, ..., \beta_9, \gamma_1, \gamma_2, ..., \gamma_9\right\}^T = \left\{0\right\}^T
\tag{16}
$$

The natural frequencies and corresponding mode shapes of beams can be calculated using equations (15) and (16). In calculating the natural frequency, the determinant of the coefficient matrix in equation (16) is computed for various values of frequency starting from a near zero value. Zero crossing of the determinant is identified and the corresponding value of frequency is the natural frequency of the beam in question.

5. Results and Discussion

In the following numerical computation, we consider the FGM beam with width *b=1 m* and thickness *h =1 m*. The position of the delamination is determined by parameters $\overline{L}_2 = L_2 / L$, $\overline{h}_2 = h_2 / \mathrm{h}$, which represent the dimensionless lengthwise and thicknesswise locations of the delamination, respectively.

Table 1 shows the non-dimensional first natural frequency ($\overline{\omega} = \omega L^2 \sqrt{I_{10} / (\mathrm{D}_{110} - \mathrm{B}_{110}^2 / \mathrm{A}_{110})}$) of an isotropic beam (by denoting $E_2 / \mathrm{E}_1 = 1$) with single of central midplane delamination of various lengths and CC boundary conditions. The results are compared with the analytical results of references [7] and [12] and FEM results of Lee [13]. The parameters $I_{10}, \mathrm{D}_{110}, \mathrm{B}_{110}, \mathrm{A}_{110}$ are the values of previously defined parameters $I_1, \mathrm{D}_{11}, \mathrm{B}_{11}, \mathrm{A}_{11}$ of an isotropic beam ($E_2 / \mathrm{E}_1 = 1$). As is shown in Table 1, the results of the current study agree well with previous published results for thin beam, but there are significant differences for thick beams.

TABLE I: Non-dimensional fundamental frequency of an isotropic beam with a central midplane delamination

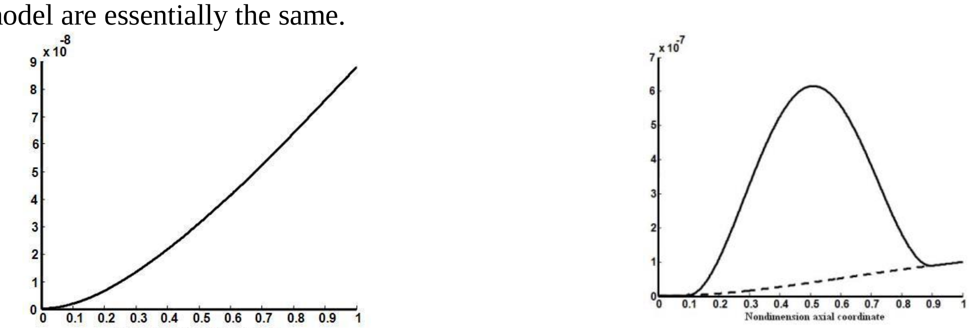
$\overline{L}_2$	Present Free and Constrained Mode						Ref. [7]	Ref. [12]	Ref. [13]
	L/h								
	5	10	15	20	40	100			
0.1	17.9207	20.9532	21.7286	21.8413	22.0434	22.3933	22.37	22.37	22.36
0.2	17.9113	20.0016	20.8333	21.2215	22.0116	22.3702	22.36	22.35	22.35
0.3	17.7835	19.5642	20.6423	21.0061	21.9867	22.2413	22.24	22.23	22.23
0.4	17.5531	19.3124	20.2187	20.6579	21.5537	21.8861	21.83	21.83	21.82

Table 2 shows the first three normalized natural frequencies ($\overline{\omega}$) of the intact FGM beams with HH boundary conditions and various slenderness ratios. This example was previously analyzed by researchers [6, 7, 14]. The natural frequencies obtained in the present study are in good agreement with previous results.

TABLE II: First three normalized natural frequencies of intact FGM beams

E_2 / E_1	Mode Number	Present						Ref. [7]	Ref. [14]	Ref. [6]
		L/h								
		5	10	15	20	40	100			
0.2	1	8.4438	8.877	8.9809	9.0225	9.1362	9.3622	9.27	9.27	9.272
	2	16.9023	21.984	23.7759	26.069	30.9294	37.0452	37.09	37.09	37.090
	3	47.1501	69.6374	74.7363	77.0345	79.856	83.3268	83.28	83.28	83.453

In Figures 2 and 3 are shown the mode shapes of the thick beams (*L/h*=15) with two different delamination lengths 0.2 and 0.8 one at the time at *k*= 0 (free mode) and CF boundary conditions. From these figures, we can see that first vibration modes do not show any opening in the cases of short delaminations and closeness to the midplane of the beam, while in cases of long delaminations and closeness to the beam surface, we can clearly see the delamination opening modes. It is clear that for modes where there is no opening in the delamination region, frequencies predicted by the free (*k*=0) and constrained (*k*=∞) modes yield the same value or are very close to each other. This is reasonable since if there is no opening in the delamination region, the free mode and constrained model are essentially the same.



(a) $\overline{L}_2 = 0.2$ (b) $\overline{L}_2 = 0.8$

Fig. 4: Mode shapes of the delaminated FGM beam with $\overline{h}_2 = 0.1, \beta = 0.2$

6. Conclusion

In the present study, an analytical solution is developed to study the free vibration of exponential functionally graded beams with a single delamination. Timoshenko beam theory, the 'free mode' and 'constrained mode' assumption in delamination vibration are adopted. The boundary conditions and continuity conditions are presented. This is the first study on the influences of delamination (its length and location) on the natural frequencies of the FGM Timoshenko beams.

7. Appendix A

By substituting equations (4) in equations (5), the B.C.s can be written as:

Clamped-Clamped Beam (CC):

$$\sum_{n=0}^{N}(-1)^n W_{1n}=0,\sum_{n=0}^{N}(-1)^n\Psi_{1n}=0,\sum_{n=0}^{N}(-1)^n U_{1n}=0$$
$$\sum_{n=0}^{N}W_{4n}=0,\ \sum_{n=0}^{N}\Psi_{4n}=0,\sum_{n=0}^{N}(-1)^n U_{4n}=0 \tag{A-1}$$

Clamped-Hinged Beam (CH):

$$\sum_{n=0}^{N}(-1)^n W_{1n}=0,\sum_{n=0}^{N}(-1)^n\Psi_{1n}=0,\ \sum_{n=0}^{N}(-1)^n U_{1n}=0$$
$$\sum_{n=0}^{N}W_{4n}=0,\left(B_{114}\sum_{n=1}^{N}U_{4n}+D_{114}\sum_{n=1}^{N}\Psi_{4n}\right)\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(2n-4k_1-1\right)=0,\sum_{n=0}^{N}(-1)^n U_{4n}=0 \tag{A-2}$$

Hinged-Hinged Beam (HH):

$$\sum_{n=0}^{N}(-1)^n W_{1n}=0,\left(B_{114}\sum_{n=1}^{N}U_{4n}+D_{114}\sum_{n=1}^{N}\Psi_{4n}\right)\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^{n-2k_1-1}\left(2n-4k_1-1\right)=0,\ \sum_{n=0}^{N}(-1)^n U_{1n}=0$$
$$\sum_{n=0}^{N}W_{4n}=0,\left(B_{114}\sum_{n=1}^{N}U_{4n}+D_{114}\sum_{n=1}^{N}\Psi_{4n}\right)\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(2n-4k_1-1\right)=0,\sum_{n=0}^{N}(-1)^n U_{4n}=0 \tag{A-3}$$

Clamped- Free Beam (CF):

$$\sum_{n=0}^{N}(-1)^n W_{1n}=0,\sum_{n=0}^{N}(-1)^n\Psi_{1n}=0,\sum_{n=0}^{N}(-1)^n U_{1n}=0$$
$$\left(A_{114}\sum_{n=1}^{N}U_{4n}+B_{114}\sum_{n=1}^{N}\Psi_{4n}\right)\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)=0,\left(B_{114}\sum_{n=1}^{N}U_{4n}+D_{114}\sum_{n=1}^{N}\Psi_{4n}\right)\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)=0,\ \sum_{n=0}^{N}\Psi_{4n}+\frac{2}{L_4}\sum_{n=1}^{N}W_{4n}\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)=0 \tag{A-4}$$

Also, by substituting equations (4) in equations (6 and 7), the C.C.s can be expressed as:

at the left end of sub-beams 2 and 3:

$$\sum_{n=0}^{N}\left[W_{1n}-(-1)^n W_{2n}\right]=0,\sum_{n=0}^{N}\left[W_{1n}-(-1)^n W_{3n}\right]=0,\sum_{n=0}^{N}\left[\Psi_{1n}-(-1)^n\Psi_{2n}\right]=0,\sum_{n=0}^{N}\left[\Psi_{1n}-(-1)^n\Psi_{3n}\right]=0,$$
$$\sum_{n=0}^{N}\left[U_{1n}-(-1)^n U_{2n}\right]-e_2\sum_{n=1}^{N}\Psi_{1n}=0,\sum_{n=0}^{N}\left[U_{1n}-(-1)^n U_{3n}\right]+e_3\sum_{n=1}^{N}\Psi_{1n}=0,$$
$$\sum_{n=0}^{N}\left[A_{551}\Psi_{1n}-A_{552}\Psi_{2n}(-1)^n-A_{553}\Psi_{3n}(-1)^n\right]+\frac{2A_{551}}{L_1}\sum_{n=1}^{N}W_{1n}\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)-\sum_{n=1}^{N}\left[\frac{2A_{552}}{L_2}W_{2n}+\frac{2A_{553}}{L_2}W_{3n}\right]\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)(-1)^{n-2k_1-1}=0,$$
$$\sum_{n=1}^{N}\left(\frac{A_{111}}{L_1}U_{1n}+\frac{B_{111}}{L_1}\Psi_{1n}\right)\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)-\sum_{n=1}^{N}\left[\frac{A_{112}}{L_2}U_{2n}+\frac{B_{112}}{L_2}\Psi_{2n}+\frac{A_{113}}{L_2}U_{3n}+\frac{B_{113}}{L_2}\Psi_{3n}\right]\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)(-1)^{n-2k_1-1}=0 \tag{A-5}$$
$$\sum_{n=1}^{N}\left(\frac{B_{111}}{L_1}U_{1n}+\frac{D_{111}}{L_1}\Psi_{1n}\right)\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)+$$
$$\sum_{n=1}^{N}\left[\left(\frac{A_{112}e_2}{L_2}-\frac{B_{112}}{L_2}\right)U_{2n}+\left(\frac{B_{112}e_2}{L_2}-\frac{D_{112}}{L_2}\right)\Psi_{2n}-\left(\frac{A_{113}e_3}{L_2}+\frac{B_{113}}{L_2}\right)U_{3n}-\left(\frac{B_{113}e_3}{L_2}+\frac{D_{113}}{L_2}\right)\Psi_{3n}\right]\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)(-1)^{n-2k_1-1}=0$$

at the right end of sub-beams 2 and 3:

$$
\begin{aligned}
&\sum_{n=0}^{N}\left[(-1)^n W_{4n}-W_{2n}\right]=0,\sum_{n=0}^{N}\left[(-1)^n W_{4n}-W_{3n}\right]=0,\sum_{n=0}^{N}\left[(-1)^n \Psi_{4n}-\Psi_{2n}\right]=0,\sum_{n=0}^{N}\left[(-1)^n \Psi_{4n}-\Psi_{3n}\right]=0,\\
&\sum_{n=0}^{N}\left[(-1)^n U_{4n}-U_{2n}\right]-e_2\sum_{n=1}^{N}(-1)^n \Psi_{4n}=0,\sum_{n=0}^{N}\left[(-1)^n U_{4n}-U_{3n}\right]+e_3\sum_{n=1}^{N}(-1)^n \Psi_{4n}=0,\\
&\sum_{n=0}^{N}\left[A_{551}(-1)^n \Psi_{4n}-A_{552}\Psi_{2n}-A_{553}\Psi_{3n}\right]+\frac{2A_{551}}{L_4}\sum_{n=1}^{N}W_{4n}\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)(-1)^{n-2k_1-1}-\sum_{n=1}^{N}\left[\frac{2A_{552}}{L_2}W_{2n}+\frac{2A_{553}}{L_2}W_{3n}\right]\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)=0,\\
&\sum_{n=1}^{N}\left(\frac{A_{111}}{L_4}U_{4n}+\frac{B_{111}}{L_4}\Psi_{4n}\right)\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)(-1)^{n-2k_1-1}-\sum_{n=1}^{N}\left[\frac{A_{112}}{L_2}U_{2n}+\frac{B_{112}}{L_2}\Psi_{2n}+\frac{A_{113}}{L_2}U_{3n}+\frac{B_{113}}{L_2}\Psi_{3n}\right]\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)=0\\
&\sum_{n=1}^{N}\left(\frac{B_{111}}{L_4}U_{4n}+\frac{D_{111}}{L_4}\Psi_{4n}\right)\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)(-1)^{n-2k_1-1}+\\
&\sum_{n=1}^{N}\left[\left(\frac{A_{112}e_2}{L_2}-\frac{B_{112}}{L_2}\right)U_{2n}+\left(\frac{B_{112}e_2}{L_2}-\frac{D_{112}}{L_2}\right)\Psi_{2n}-\left(\frac{A_{113}e_3}{L_2}+\frac{B_{113}}{L_2}\right)U_{3n}-\left(\frac{B_{113}e_3}{L_2}+\frac{D_{113}}{L_2}\right)\Psi_{3n}\right]\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)=0
\end{aligned}
\tag{A-6}
$$

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