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Piecewise Affine Modeling of a Hybrid 3-Tank System

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Abstract**:** *Hybrid systems are a fine fusion of continuous dynamical systems and discrete dynamical systems. A particular class of hybrid systems can be defined by a piecewise affine system. This paper presents Piecewise Affine (PWA) modeling of a Hybrid 3-tank system.*

Keywords: *Hybrid system, PWA modeling, 3-tank system.*

1. Introduction

Hybrid systems are systems where both the continuous and the discrete dynamics co-exist and interact with each other, that is they freely combine dynamical features from both these systems and are therefore claimed to be more challenging that the continuous dynamical systems.[1]. The fundamental feature is its nature of hybridness that enhances flexibility and proves perfect combination of discrete and continuous reasoning. For describing such hybrid systems, a class of powerful modelling framework known as piecewise affine (PWA) systems are used. [2]. Wherein, the complex dynamics of continuous model is replaced by piecewise affine approximations which allow an analytical solution. The sequence of affine models then forms a sequence of states of a hybrid automaton.

2. Piecewise affine systems

PWA systems are amongst the most studied forms of hybrid systems. A PWA system is defined as; [3]

$$x(t+1) = A^i x(t) + B^i u(t) + f^i \qquad \text{for} \quad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i \tag{1}$$

$$y(t) = C^i x(t) + g^i$$

Where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^r$ denote the state, input and output vectors, respectively. $\{\mathcal{X}_i\}_{i=1}^{s}$ is a polyhedral partition of the states and input space. Each $\mathcal{X}_i$ is given by:

$$\mathcal{X}_i \triangleq \left\{ \begin{bmatrix} x(t) \\ U(t) \end{bmatrix} | R^i \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \leq r^i \tag{2}$$

Each subsystem $\mathbb{S}^i$ defined by the 7-tuple $(A^i, B^i, C^i, f^i, g^i, R^i, r^i)$, $i \in \{1,2, \ldots\ldots s\}$, is termed as a component of the PWA system (1). $A^i \in \mathbb{R}^{n\times n}$, $B^i \in \mathbb{R}^{n\times m}$, and (A^i, B^i) is a controllable pair. $C^i \in \mathbb{R}^{r\times n}$ and $R^i \in \mathbb{R}^{P_i \times (n\times m)}$ and f^i, g^i, r^i are suitable constant vectors. Note that *n* is the number of states, *m* is the number of inputs, *r* is the number of outputs, and p_i is the number of hyperplanes that define $\mathcal{X}_i$.

PWA systems are equivalent to sets of linear systems combined with finite automatons. On the other hand, PWA systems allow to model a wide class of processes such as linear systems with static piecewise affine nonlinearities or commuted systems. Moreover, they are able to approximate nonlinear dynamics arbitrarily well using local linearization around different working points, and also they can approximate the more general type of nonlinear hybrid systems replacing the nonlinearities by piecewise affine approximations. They are also suitable for stability analysis.

Another interesting property of PWA systems is their equivalence to other types of hybrid systems models such as linear complementary systems, extended linear complementary systems, max-min-plus-scaling systems or the more popular mixed logical dynamical (MLD) systems.[4] All these models have their advantages, but the equivalence among all of them allow to transfer theoretical results from one type to other. In this regard, PWA systems are by far the most studied amongst them.

3. Hybrid 3-tank system

The levels of water in tanks and outflows constitute the analog part, and valves between tanks, which can be either opened or closed (taking respective values of 1 and 0) are the discrete part.

The three tank system depicted in Figure 1. has been adopted. The system consists of three liquid tanks that can be filled with two identical, independent pumps acting on the outer tanks 1 and 2. The pumps deliver the liquid flows q1 and q2 and they can be continuously manipulated from a minimum flow to a maximum flow. The tanks are interconnected to each other through pipes. The flow through these pipes can be interrupted with switching valves V_{13}, V_{23} that can assume either the completely open or the completely closed position. The liquid levels h_1, h_2, h_3 in each tank can be measured with continuous valued level sensors. The nominal outflow from the system is located at the middle tank, i.e. V_3 is always open.

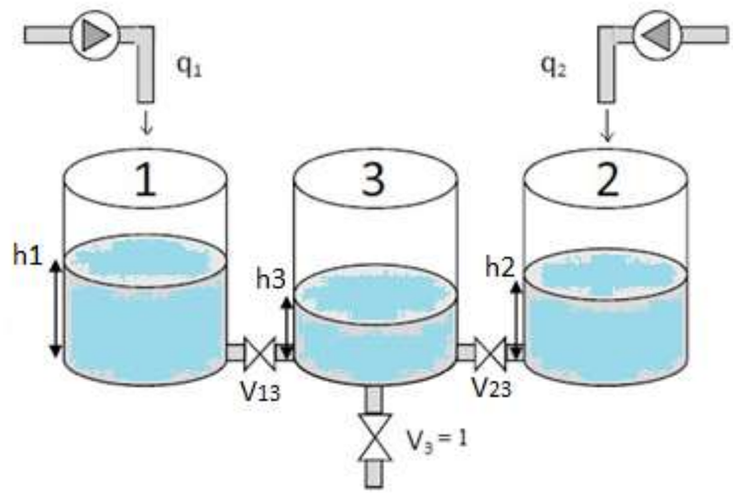

Fig. 1: Three tank system.

The mathematical model of the three tank system is shown in Eq (3) to (5).

$$\frac{dh_1}{dt} = \frac{q_1}{S_1} - sign(h_1 - h_3)V_{13}\frac{S_{13}}{S_1}\sqrt{|2g(h_1 - h_3)|} - V_1\frac{S_{01}}{S_1}\sqrt{2gh_1} \tag{3}$$

$$\frac{dh_2}{dt} = \frac{q_2}{S_2} - sign(h_2 - h_3)V_{23}\frac{S_{23}}{S_2}\sqrt{|2g(h_2 - h_3)|} - V_2\frac{S_{02}}{S_2}\sqrt{2gh_2} \tag{4}$$

$$\frac{dh_3}{dt} = sign(h_1 - h_3)V_{13}\frac{S_{13}}{S_3}\sqrt{|2g(h_1 - h_3)|} + sign(h_2 - h_3)V_{23}\frac{S_{23}}{S_3}\sqrt{|2g(h_2 - h_3)|} - V_3\frac{S_{03}}{S_3}\sqrt{2gh_3} \tag{5}$$

where:

- h_1, h_2, h_3 – levels in respective tanks
- S_1,S_2,S_3 – cross sections of tanks (tanks dimensions are equal (=S))
- S_{13}, S_{23} – cross section of digital valves between tanks
- S_{01}, S_{02} – cross section of Valves
- q_1, q_2 – inflow through pumps
- V_{13},V_{23} – status of digital valves between tanks (0-closed, 1-opened)

- V_1, V_2, V_3 – status of output valves(0-closed, 1-opened)
- g – gravitational acceleration

The model of the 3-tank system given in Eq(3) to (5) is further modified for a particular configuration of the 3-tank system with the assumption that $h_1 > h_3$ and $h_2 > h_3$. Eq. (6) to (8) represents the model of 3-tank system for such a configuration.

$$\frac{dh_1}{dt} = \frac{q_1}{S} - V_{13}\frac{S_{i3}}{S}\sqrt{2g(h_1 - h_3)} \quad (6)$$

$$\frac{dh_2}{dt} = \frac{q_2}{S} - V_{23}\frac{S_{i3}}{S}\sqrt{2g(h_2 - h_3)} \quad (7)$$

$$\frac{dh_3}{dt} = V_{13}\frac{S_{i3}}{S}\sqrt{2g(h_1 - h_3)} + V_{23}\frac{S_{i3}}{S}\sqrt{2g(h_2 - h_3)} - V_3\frac{S_{oi}}{S}\sqrt{2gh_3} \quad (8)$$

Linearized Equations take the form:-

$$\frac{d\Delta h_1}{dt} = \frac{\Delta q1}{s} - (V_{13}\frac{S_{i3}}{S}\frac{g}{\sqrt{2g(h_1-h_3)}}\Delta h_1 - V_{13}\frac{S_{i3}}{S}\frac{g}{\sqrt{2g(h_1-h_3)}}\Delta h_3) \quad (9)$$

$$\frac{d\Delta h_2}{dt} = \frac{\Delta q2}{s} - (V_{23}\frac{S_{i3}}{S}\frac{g}{\sqrt{2g(h_2-h_3)}}\Delta h_2 - V_{23}\frac{S_{i3}}{S}\frac{g}{\sqrt{2g(h_2-h_3)}}\Delta h_3) \quad (10)$$

$$\frac{d\Delta h_3}{dt} = (V_{13}\frac{S_{i3}}{S}\frac{g}{\sqrt{2g(h_1-h_3)}}\Delta h_1 - V_{13}\frac{S_{i3}}{S}\frac{g}{\sqrt{2g(h_1-h_3)}}\Delta h_3) + (V_{23}\frac{S_{i3}}{S}\frac{g}{\sqrt{2g(h_2-h_3)}}\Delta h_2 - V_{23}\frac{S_{i3}}{S}\frac{g}{\sqrt{2g(h_2-h_3)}}\Delta h_3) - (-\frac{S_i}{S}\frac{g}{\sqrt{2gh_3}}\Delta h_3) \quad (11)$$

Linearization is done around a operating point. The operating points chosen are $h_1 = 60$, $h_2 = 40$ and $h_3 = 20$.

The State-Space Matrices are computed as below:-

$$A=\begin{bmatrix} -C_{13} & 0 & C_{13} \\ 0 & -C_{23} & C_{23} \\ C_{13} & C_{23} & -C_{13}-C_{23}-C_3 \end{bmatrix},\quad B=\begin{bmatrix} \frac{1}{S} & 0 \\ 0 & \frac{1}{S} \\ 0 & 0 \end{bmatrix},\quad C=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad D=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where:

$$C_{13} = V_{13}\frac{S_{i3}}{S}\frac{g}{\sqrt{2g(h_{10}-h_{30})}} \quad (12)$$

$$C_{23} = V_{23}\frac{S_{i3}}{S}\frac{g}{\sqrt{2g(h_{20}-h_{30})}} \quad (13)$$

$$C_3 = \frac{S_{oi}}{S}\frac{g}{\sqrt{2gh_{30}}} \quad (14)$$

4. Results

Figure 2, 3 4 and 5 show the state space partition for the valve (V_{13}, V_{23}) conditions (1, 1), (0, 1), (1, 0) and (0, 0) respectively. It is evident that the state variables have different values for different binary conditions of the valves. Figure 6 show the plot for the complete (combination of figure 2, 3, 4 and 5) state space partition of the hybrid 3-tank system. x_1, x_2 and x_3 are the state variables. They represent the height of the liquid in tank 1, tank 2 and tank 3 respectively in millimeters at a given point of time.

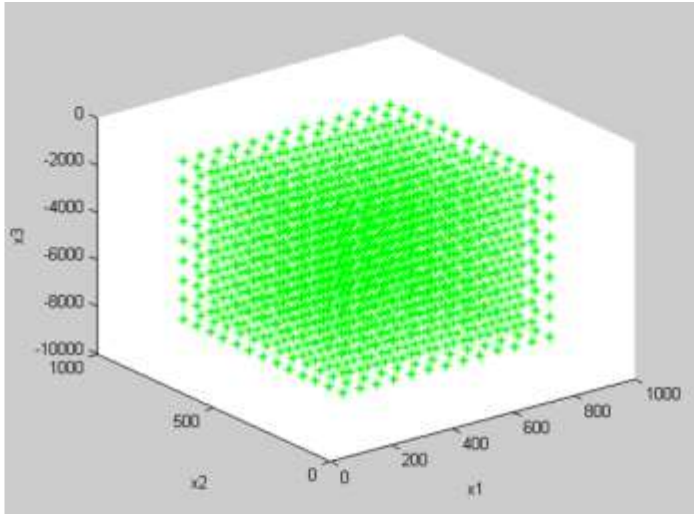

Fig. 2. State space partition for the valve condition (1, 1)

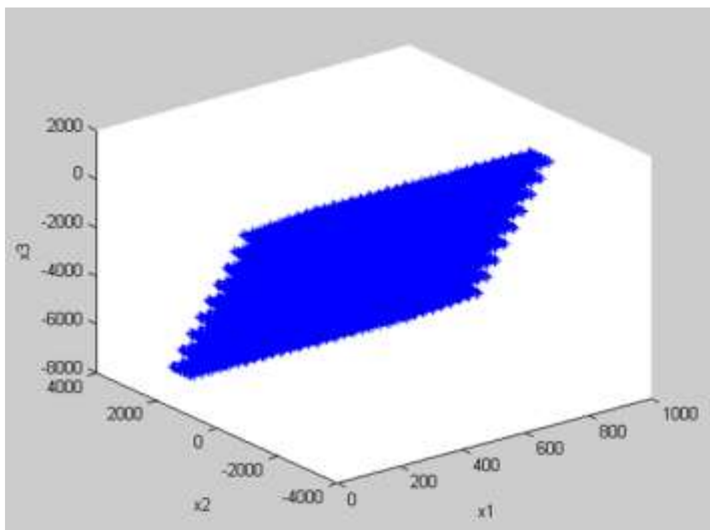

Fig. 3. State space partition for the valve condition (0, 1)

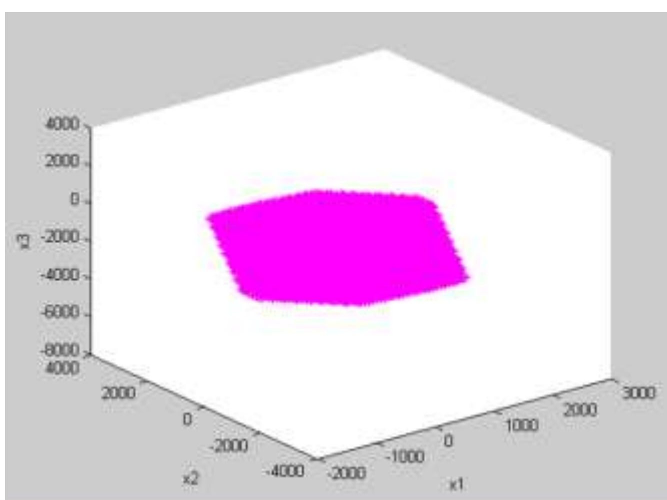

Fig. 4. State space partition for the valve condition (1, 0)

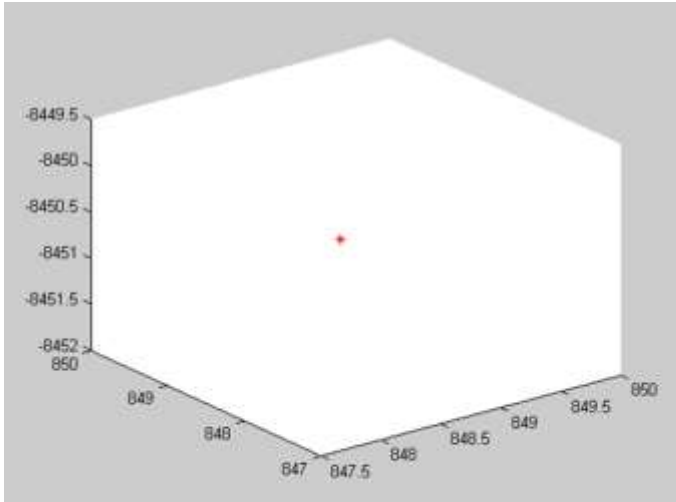

Fig. 5: State space partition for the valve condition (0, 0)

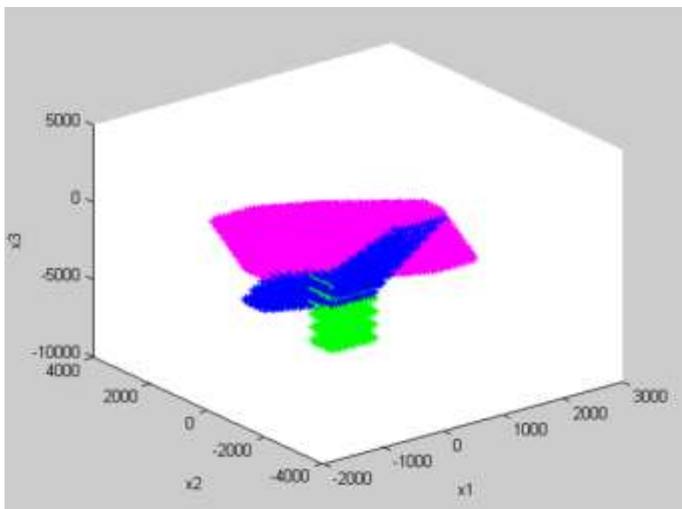

Fig. 6: State space partition for the 3-tank system

5. Conclusion

The piecewise affine model of the hybrid 3-tank system was implemented in Matlab. It is evident from the results that the state space of a hybrid system has different partitions for different discrete conditions based on the discrete components in the hybrid system. The approach gives a better platform for designing a hybrid controller for the hybrid system. The PWA model can be used to design a Model Predictive controller for the hybrid 3 tank system.

6. References

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